PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Forced Flow due to Heating of a Rotating Liquid

T. V. Davies

Phil. Trans. R. Soc. Lond. A 1956 **249**, 27-64
doi: 10.1098/rsta.1956.0014

THE FORCED FLOW DUE TO HEATING OF A ROTATING LIQUID

By T. V. Davies, *King's College, London*

(*Communicated by G. Temple, F.R.S.—Received* 14 *June* 1955)

CONTENTS

An investigation is made of the forced liquid motion in a rotating cylindrical vessel with a horizontal base when a temperature difference exists between the outer and inner cylindrical boundaries of the liquid. It has been observed experimentally that at critical values of a certain non-dimensional parameter, known as the Rossby number, the flow patterns change abruptly in character. The present investigation derives a stability criterion which agrees qualitatively with the experimental results and also gives reasonable quantitative results. In the derivation of this result it is shown that there exists a relation between the mean vertical temperature gradient, the mean horizontal temperature gradient and the angular velocity of rotation of the system.

1. Introduction

The dishpan experiments by Fultz in Chicago and Hide's heated rotating cylinder experiment at Cambridge have been described by Fultz (1953), and the importance of the experiments to meteorology has been established by Starr & Long's (1953) investigation of the transfer of momentum. Theoretical work which has as its aim the understanding of the thermodynamical and dynamical processes has been initiated by Kuo (1955), Lorenz (1953) and by the present author (1953). In this paper it is proposed to investigate theoretically what has become known as the Rossby régime in these experiments which is the régime of principal interest to meteorology.

In the typical experiment a shallow liquid of suitable low viscosity is contained in a cylindrical vessel, or in the annulus between two concentric cylinders, the usual dimensions in the former case being a radius of 15 cm and a liquid depth of about 3 cm. The effects of heating the horizontal base of the vessel near the side have been observed for various rates of rotation of the vessel about its central axis. The heating in such experiments is approximately symmetrical about the central axis, and the difference in temperature between the outer portions of the liquid and the central portions is usually between 5 and 15° C. In Hide's experiment and in recent experiments by Fultz the outer and inner cylinders are maintained at constant temperatures, T_0 and T_i ($T_0 > T_i$) respectively, and the observed flow patterns are then considerably more steady and more controlled. Two principal régimes have emerged which are referred to as the Hadley or spiral régime or low-rotation régime and the Rossby or wave régime or high-rotation régime. In the former the motion near the free upper surface is predominantly symmetrical about the central axis, and the stream-lines and the paths of the particles spiral inwards towards the central axis. Within the liquid and near the base of the vessel the liquid spirals outwards, and in the case of two concentric cylinders this outward spiral persists at all distances from the central axis. When the vessel rotates from west to east, as in the case of the earth's rotation, there is a deflexion to the right as in 'Ferrel's rule', so that the inflowing spiral at the top winds counterclockwise when viewed from above and the outflowing spiral near the base winds clockwise. In the case of one outer cylinder and liquid occupying the whole of the interior, the outflowing spiral near the base does not wind clockwise for all radii, and it has been shown by Fultz that in this case the flow near the outer cylindrical wall is in the reverse direction. This reversal is necessary in order to preserve the angular momentum balance and is a feature of the mathematical solution investigated by the present author (1953). In the Rossby régime, on the other hand, the motion is markedly asymmetrical in character, and the stream-line pattern of the free upper surface, observed relative to the rotating vessel, consists of distinct finite-amplitude wave patterns which progress slowly relative to the vessel in the same direction as the rotation. With a fixed heating system at the base or with a constant maintained temperature difference between the outer and inner cylinders, the transition from the Hadley to the Rossby régime occurs at a critical angular velocity of rotation Ω_c of the vessel. The range $0<\Omega<\Omega_c$ of the angular velocity then corresponds to the Hadley régime and $\Omega>\Omega_c$ to the Rossby régime. When Ω is just slightly above Ω_c the number of wave petals in the stream-line pattern is usually small, and as Ω increases from this value the number of petals increases, a three-wave pattern, say, giving way to a four-wave pattern and so on. This is evidently a stability problem, and one of the aims of this investigation is to seek to explain why the spiral régime becomes unstable at a critical angular velocity and why an m-wave régime moves over to an $(m+1)$-wave régime at a different critical value. Qualitatively, it is fairly clear that the spiral régime cannot exist indefinitely as Ω is increased; for with increasing rotation there is a tightening of the spiral in the inflow towards the central axis in the upper portion of the liquid, and thus the efficiency of the liquid as a convector of heat steadily diminishes; therefore, a critical stage necessarily arises when the liquid chooses a new method of convecting the heat. In addition to the stability problem, the theory must predict a formula for the angular velocity of the waves relative to the rotating vessel which must bear comparison with certain empirical formulae derived by Hide (1953).

The three-dimensional structure of the wave patterns in the Rossby régime have now been explored to a certain extent, and it would appear that the structure is similar to the long atmospheric waves in that the liquid which is moving inward (towards the central axis) is ascending while the outward moving liquid is descending. In addition, there is a phase shift of the waves in the vertical direction.

One feature of the experiments which cannot be investigated here is that called 'vacillation' by Hide. This is an oscillation of the long wave about its original horizontal axis of symmetry which occurs when the amplitude/wavelength ratio exceeds a critical quantity 0·67. This is probably a non-linear phenomenon, and the present investigation, which is entirely 'linear', cannot hope to cast any light upon the feature.

In their discussion of the experimental results both Fultz and Hide adequately express their results in terms of a single parameter called the Rossby number, which is introduced in the present paper in (1·18) where it is denoted by R_H. Fultz (1953) uses the Rossby number Ro_T which is identical with the present R_H, and broadly his experimental results can be summarized as follows. The Hadley or spiral régime is 'associated with values of Ro_T in general greater than some limit in the interval 0·2 to 0·6' and 'in the Rossby régime Ro_T has values of the order of 0·1'. In general, higher wave numbers appear as Ro_T (or R_H) is decreased although experimentally there is an overlap of the Ro_T intervals in which two neighbouring wave numbers can exist. Some of the experimental figures quoted by Fultz (1953) are incorporated in table 5 later in this paper, and there is a further discussion of the experimental results also towards the end of this paper.

In setting up the problem, which is essentially three-dimensional, I use cylindrical coordinates (r, ϕ, z), and there are six dependent variables $(u_1, v_1, w_1, p_1, \rho_1, T_1)$, where u_1 represents the velocity component in the direction r increasing, v_1 in the direction ϕ increasing, w_1 in the direction z increasing, where p_1 is the pressure, ρ_1 the density and T_1 the temperature. Connecting these six dependent variables are the following six equations:

$$\rho_1\left(\frac{\mathrm{d}u_1}{\mathrm{d}t}-\frac{v_1^2}{r}\right) = -\frac{\partial p_1}{\partial r}+\mu\left(\nabla_1^2 u_1-\frac{u_1}{r^2}-\frac{2}{r^2}\frac{\partial v_1}{\partial \phi}\right)+\tfrac{1}{3}\mu\frac{\partial \chi_1}{\partial r} = -\frac{\partial p_1}{\partial r}+\mu F_r, \tag{1·1}$$

$$\rho_1\left(\frac{\mathrm{d}v_1}{\mathrm{d}t}+\frac{u_1 v_1}{r}\right) = -\frac{\partial p_1}{r\,\partial \phi}+\mu\left(\nabla_1^2 v_1-\frac{v_1}{r^2}+\frac{2}{r^2}\frac{\partial u_1}{\partial \phi}\right)+\tfrac{1}{3}\mu\frac{\partial \chi_1}{r\,\partial \phi} = -\frac{\partial p_1}{r\,\partial \phi}+\mu F_\phi, \tag{1·2}$$

$$\rho_1\frac{\mathrm{d}w_1}{\mathrm{d}t} = -\frac{\partial p_1}{\partial z}-g\rho_1+\mu\nabla_1^2 w_1+\tfrac{1}{3}\mu\frac{\partial \chi_1}{\partial z} = -\frac{\partial p_1}{\partial z}-g\rho_1+\mu F_z, \tag{1·3}$$

$$\frac{\mathrm{d}\rho_1}{\mathrm{d}t}+\rho_1\chi_1 = 0, \quad \chi_1 = \frac{\partial u_1}{\partial r}+\frac{u_1}{r}+\frac{\partial v_1}{r\,\partial \phi}+\frac{\partial w_1}{\partial z}, \tag{1·4}$$

$$f(\rho_1, T_1) = 0, \tag{1·5}$$

$$\rho_1 J c_v \frac{\mathrm{d}T_1}{\mathrm{d}t} = Jk\nabla_1^2 T_1+\Phi_1, \tag{1·6}$$

where

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}. \tag{1·7}$$

Equations (1·1) to (1·3) are the equations of motion, (1·4) is the equation of continuity, (1·5) is the equation of state for a liquid and (1·6) is the equation of heat transfer for a liquid. We consider first of all the equation of state which, within the temperature range of the experiment, can be chosen to be a linear relation between density and temperature. One of the important features of the experiment is that when the mean temperature field of all the planes ϕ = constant is obtained it reveals that there is an increase of temperature from the base to the free surface of the liquid and an increase of temperature from the central axis (or inner cylinder) to the outer cylindrical wall. It is quite evident that we may regard the temperature pattern in any ϕ plane to be a small departure from this mean temperature pattern. Accordingly, we shall write the density and temperature in the forms

$$\rho_1 = \rho^*(r, z)+\rho(r, \phi, z, t), \tag{1·8}$$

$$T_1 = T^*(r, z)+T(r, \phi, z, t), \tag{1·9}$$

where ρ^* and T^* represent the mean meridional fields of density and temperature respectively and are functions of r, z only, while ρ and T represent the small departures from the mean values. Using the linearity of the relation between density and temperature, we thus have the relations

$$\rho^* = \rho_0 - \alpha(T^* - T_0), \tag{1·10}$$

$$\rho = -\alpha T, \tag{1·11}$$

where ρ_0 is the constant density of the liquid at the temperature T_0 and α is the inverse of the coefficient of cubical expansion. Using the results quoted in Davies (1953), the appropriate value of α for the range 20 to 30° C is $\alpha = 2{\cdot}5563 \times 10^{-4}$.

Owing to the presence of this mean temperature field there will be established a mean zonal motion, in the ϕ-increasing direction, relative to the rotating cylinder. Accordingly, we shall assume that the complete velocity field is given by

$$u_1 = u(r, \phi, z, t), \quad v_1 = r\Omega + V_0(r, z) + v(r, \phi, z, t), \quad w_1 = w(r, \phi, z, t), \tag{1·12}$$

where $V_0(r, z)$ is the mean zonal flow relative to the cylinder which rotates with an angular velocity Ω. The expression for the pressure will be

$$p_1 = P_0(r, z) + p(r, \phi, z, t), \tag{1·13}$$

where P_0 is the mean pressure arising from the mean zonal flow and hydrostatic sources.

The equations which govern the mean zonal flow patterns are as follows:

$$\left.\begin{aligned} -\rho_0(r\Omega + V_0)^2/r &= -\frac{\partial P_0}{\partial r}, && (a)\\ 0 &= \nabla_1^2 V_0 - \frac{V_0}{r^2}, && (b)\\ 0 &= -\frac{\partial P_0}{\partial z} - g\rho^*, && (c)\\ \rho^* &= \rho_0 - \alpha(T^* - T_0), && (d)\\ 0 &= Jk\nabla_1^2 T^* + \Phi_0. && (e)\end{aligned}\right\} \tag{1·14}$$

In (1·14 e) the function Φ_0 which represents the dissipation of energy due to molecular viscosity is here given by

$$\Phi_0 = \mu\left\{\left(\frac{\partial V_0}{\partial r} - \frac{V_0}{r}\right)^2 + \left(\frac{\partial V_0}{\partial z}\right)^2\right\},$$

and in (1·14 a) it will be noted that the constant density ρ_0 of the liquid appears on the left-hand side in place of the exact ρ^*. Since the difference is small, equation (1·14 a) is sufficiently exact in this form. It is not practicable to seek or to use any exact solution of the equations (1·14) for the resulting equations for the perturbation field become completely intractable in this case. We postulate therefore a basic temperature field in the form

$$T^* = T_0 + \Theta_V z + \tfrac{1}{2}r^2\Theta_H, \tag{1·15}$$

where Θ_V and Θ_H are positive constants. A recent set of temperature measurements by Fultz in the case of a three-wave pattern between concentric cylinders gives some idea of the magnitude of these constants, these being $\Theta_V = 1{\cdot}45°/\text{cm}$, $\Theta_H = 0{\cdot}16°/\text{cm}^2$. It is important here to emphasize the positive sign of Θ_V, and it is necessary to bear in mind that in making

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

any comparisons with meteorology the quantity that corresponds to T^* is the potential temperature, which increases in value in the vertical direction. In order to derive the velocity field arising from (1·15) we write (1·14*a*) in the approximate form

$$\rho_0(r\Omega^2+2\Omega V_0)=\frac{\partial P_0}{\partial r},$$

since V_0 is everywhere considerably less than $2r\Omega$ at the corresponding value of r. It then follows that

$$2\Omega\rho_0\frac{\partial V_0}{\partial z}=g\alpha\frac{\partial T^*}{\partial r}, \tag{1·16}$$

and thus the solution for V_0 which satisfies (1·14*b*) and the condition $V_0=0$ at $z=0$ is

$$V_0=2\Omega R_H(z/h)\,r, \tag{1·17}$$

where h is the depth of the liquid in the experiment and R_H is a non-dimensional constant defined by

$$R_H=\frac{g\alpha h\Theta_H}{4\rho_0\Omega^2}. \tag{1·18}$$

The velocity field (1·17) cannot be correct for all values of r, since at the boundary $r=r_0$, V_0 actually vanishes, but within a region $0\leqslant r\leqslant r_b<r_0$, which includes the central axis, it can be accepted as an approximation to the mean velocity field (see table 1). An alternative procedure would be to choose V_0 to be that zonal flow which is derived in Davies (1953), so that $V_0\propto J_1(\beta r)$, which is known to represent the zonal field reasonably well; this, however, makes the perturbation equations intractable. With the temperature field (1·15) and the velocity field (1·17), equation (1·14*e*) cannot be satisfied, and it is necessary to supply heat to the liquid in order to maintain this distribution. Although many of the assumptions introduced in this basic flow are rather crude, the important feature of 'baroclinity', which is the meteorological term for the non-coincidence of the isobaric and isothermal surfaces, is certainly incorporated in the above basic fields through the constant Θ_H, and we shall later find that the non-dimensional parameter R_H in (1·18), which depends upon Θ_H, plays an important role throughout the work. Both Fultz and Hide introduce a non-dimensional parameter similar to (1·18); this will be discussed at greater length later in the paper. It may be observed here, however, that R_H is a type of Richardson number although not in the conventional sense which refers to density and velocity gradients in the vertical; it is later interpreted as the Rossby number of the problem.

TABLE 1. MEAN VALUES OF V_0 MEASURED ON THE FREE SURFACE AT VARIOUS RADIAL DISTANCES r MEASURED FROM THE CENTRAL AXIS. (AFTER STARR & LONG)

V_0 (cm/s)	0·0705	0·1413	0·2449	0·4119	0·7290	0·6375
r (cm)	2·50	5·00	7·50	10·00	12·50	13·75

The equations which govern the perturbation field are now assumed to be the linearized form of the set (1·1) to (1·6) when products and squares of the terms u, v, w, p, ρ and T are neglected. The three equations of motion which result are then found to be as follows for the motion *relative* to the rotating cylinder, in which ϕ now refers to longitude relative to a fixed radius in the cylinder:

$$\rho_0\left\{\left(\frac{\partial}{\partial t}+\frac{V_0}{r}\frac{\partial}{\partial\phi}\right)u-v\left(2\Omega+\frac{2V_0}{r}\right)\right\}=-\frac{\partial p}{\partial r}+\mu\left\{\nabla_1^2u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial\phi}+\frac{1}{3}\frac{\partial\chi}{\partial r}\right\}, \tag{1·19}$$

$$\rho_0\left\{\left(\frac{\partial}{\partial t}+\frac{V_0}{r}\frac{\partial}{\partial\phi}\right)v+u\left(2\Omega+\frac{V_0}{r}+\frac{\partial V_0}{\partial r}\right)\right\}=-\frac{\partial p}{r\partial\phi}+\mu\left\{\nabla_1^2v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial\phi}+\frac{1}{3}\frac{\partial\chi}{r\partial\phi}\right\}, \tag{1·20}$$

$$\rho_0\left(\frac{\partial}{\partial t}+\frac{V_0}{r}\frac{\partial}{\partial\phi}\right)w=-\frac{\partial p}{\partial z}+g\alpha T+\mu\left(\nabla_1^2w+\frac{1}{3}\frac{\partial\chi}{\partial r}\right), \tag{1·21}$$

where χ is the divergence defined by

$$\chi=\frac{\partial u}{\partial r}+\frac{u}{r}+\frac{\partial v}{r\partial\phi}+\frac{\partial w}{\partial z}. \tag{1·22}$$

It will be observed that the density in these equations has been replaced everywhere by its constant value ρ_0 at temperature T_0 except, of course, in the buoyancy term, where it is replaced by the value $-\alpha T$ from (1·11). Furthermore, in (1·20) the term $w\,\partial V_0/\partial z$ has been ignored in comparison with the other terms on the left-hand side which are horizontal velocity terms; it will be evident later that this approximation is fully justified. In the equations of continuity and heat transfer a new type of approximation is incorporated in the following way. We have

$$\frac{\mathrm{d}T_1}{\mathrm{d}t}=\left(\frac{\partial}{\partial t}+\frac{V_0+v}{r}\frac{\partial}{\partial\phi}+u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)(T^*+T)$$
$$=\frac{\partial T}{\partial t}+\frac{V_0}{r}\frac{\partial T}{\partial\phi}+u\frac{\partial T^*}{\partial r}+w\frac{\partial T^*}{\partial z}+\text{second-order terms.}$$

Using (1·15) we may here replace $\partial T^*/\partial z$ by Θ_V and $\partial T^*/\partial r$ by $r\Theta_H$. If the resulting equations are considered with this operator in the heat-transfer equation it is found that they are quite intractable, since the method of separation of variables is not applicable. It is proposed, therefore, to replace the velocity component u by its 'geostrophic value' in the unsteady case. Precisely what this implies will be seen in the next section, and here it is sufficient to state that ru is to be replaced by $-F(z)\,\partial p/\partial\phi$, and accordingly we may write the above expression in the form

$$\frac{\mathrm{d}T_1}{\mathrm{d}t}=\left(\frac{\partial}{\partial t}+\frac{V_0}{r}\frac{\partial}{\partial\phi}\right)T-\Theta_HF(z)\frac{\partial p}{\partial\phi}+\Theta_Vw+\text{second-order terms,} \tag{1·23}$$

and in this case the equations of continuity and heat transfer become

$$\alpha\left\{\left(\frac{\partial}{\partial t}+\frac{V_0}{r}\frac{\partial}{\partial\phi}\right)T-\Theta_HF(z)\frac{\partial p}{\partial\phi}+\Theta_Vw\right\}=\rho_0\chi \tag{1·24}$$

and
$$\rho_0c_v\left\{\left(\frac{\partial}{\partial t}+\frac{V_0}{r}\frac{\partial}{\partial\phi}\right)T-\Theta_HF(z)\frac{\partial p}{\partial\phi}+\Theta_Vw\right\}=k\nabla_1^2T. \tag{1·25}$$

We now have five equations, namely, (1·19), (1·20), (1·21), (1·24) and (1·25), between the five dependent variables u, v, w, p, T, and these equations form a consistent set.

2. The method of solution of the perturbation equations

We first of all assume that the dependent variables are of the form

$$(u, v, w, p, T, \chi) = (u, v, w, p, T, \chi) \exp(\mathrm{i}m\phi + \mathrm{i}\sigma t), \tag{2·1}†$$

where m is the wave number of the perturbation and σ its frequency. The five equations governing the perturbation motion are then

$$\mathrm{i}\rho_0 \sigma' u - 2\rho_0 \Omega' v = -\frac{\partial p}{\partial r} + \gamma \frac{\partial \chi}{\partial r} + \mu\left(\nabla^2 u - \frac{u}{r^2} - \frac{2\mathrm{i}mv}{r^2}\right), \tag{2·2}$$

$$\mathrm{i}\rho_0 \sigma' v + 2\rho_0 \Omega' u = -\frac{\mathrm{i}mp}{r} + \frac{\mathrm{i}m\gamma\chi}{r} + \mu\left(\nabla^2 v - \frac{v}{r^2} + \frac{2\mathrm{i}mu}{r^2}\right), \tag{2·3}$$

$$\mathrm{i}\rho_0 \sigma' w = -\frac{\partial p}{\partial z} + \gamma \frac{\partial \chi}{\partial z} + g\alpha T + \mu \nabla^2 w, \tag{2·4}$$

$$\alpha\{\mathrm{i}\sigma' T - \mathrm{i}m\Theta_H F(z)\, p + \Theta_V w\} = \rho_0 \left\{\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\mathrm{i}mv}{r} + \frac{\partial w}{\partial z}\right\} = \rho_0 \chi, \tag{2·5}$$

$$\rho_0 c_v \{\mathrm{i}\sigma' T - \mathrm{i}m\Theta_H F(z)\, p + \Theta_V w\} = k \nabla^2 T, \tag{2·6}$$

where $\gamma = \frac{1}{3}\mu$,

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} - \frac{m^2}{r^2}, \tag{2·7}$$

and where σ' and Ω' are defined by

$$\left.\begin{aligned} \sigma' &= \sigma + 2\Omega R_H(z/h)\, m, \\ \Omega' &= \Omega + 2\Omega R_H(z/h). \end{aligned}\right\} \tag{2·8}$$

It is of interest to note at this stage that the incorporation of the 'baroclinicity' through the term R_H has the effect of making both the frequency term σ' and the Coriolis term Ω' depend upon z, so that the effective frequency and effective angular velocity will vary from one level to the next.

The next step consists in separating the variables r and z in equations (2·2) to (2·6). This was done originally by recasting the equations in such a way that $u+\mathrm{i}v$ and $u-\mathrm{i}v$ replace u and v as dependent variables. The detailed method need not be explained, and it is sufficient to state that when two new variables $\xi(z)$ and $\eta(z)$ are introduced so that

$$2u = \xi(z)\frac{mC_m(\beta r)}{\beta r} - \eta(z)\, C'_m(\beta r), \tag{2·9}$$

$$2\mathrm{i}v = -\xi(z)\, C'_m(\beta r) + \eta(z)\frac{mC_m(\beta r)}{\beta r}, \tag{2·10}$$

$$w = W(z)\, C_m(\beta r), \tag{2·11}$$

$$p = P(z)\, C_m(\beta r), \tag{2·12}$$

$$T = \tau(z)\, C_m(\beta r), \tag{2·13}$$

$$\chi = \chi_0(z)\, C_m(\beta r), \tag{2·14}$$

where $C_m(\varpi)$ is a solution of Bessel's differential equation of order m

$$\frac{\mathrm{d}^2 C_m}{\mathrm{d}\varpi^2} + \frac{1}{\varpi}\frac{\mathrm{d}C_m}{\mathrm{d}\varpi} + \left(1 - \frac{m^2}{\varpi^2}\right) C_m = 0, \tag{2·15}$$

† No confusion will arise from the use of the same variables on each side of this equation.

the variables r and z in (2·2) to (2·6) become separable. The five resulting equations for $\xi(z)$, $\eta(z)$, $W(z)$, $P(z)$ and $\tau(z)$ are then the following five ordinary simultaneous equations:

$$\mathrm{i}\rho_0\sigma'\xi + 2\mathrm{i}\rho_0\Omega'\eta = \mu\left(\frac{\mathrm{d}^2\xi}{\mathrm{d}z^2} - \beta^2\xi\right), \tag{2·16}$$

$$\mathrm{i}\rho_0\sigma'\eta + 2\mathrm{i}\rho_0\Omega'\xi = 2\beta(P - \gamma\chi_0) + \mu\left(\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2} - \beta^2\eta\right), \tag{2·17}$$

$$\mathrm{i}\rho_0\sigma' W - g\alpha\tau = -\left(\frac{\mathrm{d}P}{\mathrm{d}z} - \gamma\frac{\mathrm{d}\chi_0}{\mathrm{d}z}\right) + \mu\left(\frac{\mathrm{d}^2 W}{\mathrm{d}z^2} - \beta^2 W\right), \tag{2·18}$$

$$\alpha\{\mathrm{i}\sigma'\tau - \mathrm{i}m\Theta_H F(z)\,P + \Theta_V W\} = \tfrac{1}{2}\rho_0\beta\eta + \rho_0\frac{\mathrm{d}W}{\mathrm{d}z} = \rho_0\chi_0, \tag{2·19}$$

$$c_v\rho_0\{\mathrm{i}\sigma'\tau - \mathrm{i}m\Theta_H F(z)\,P + \Theta_V W\} = k\left(\frac{\mathrm{d}^2\tau}{\mathrm{d}z^2} - \beta^2\tau\right). \tag{2·20}$$

In their present form these five equations apply to a wide variety of heating problems in cylindrical co-ordinates.

It is of interest to note at this stage that the function $C_m(\beta r)$ is any solution of (2·15), so that for a liquid bounded by one outer cylindrical wall, we may utilize the function $J_m(\beta r)$, which is finite at $r = 0$, and for a liquid between two concentric cylinders we may utilize the function $\alpha_m J_m(\beta r) + \beta_m Y_m(\beta r)$. In each case the resulting form of the equations (2·16) to (2·20) is unchanged. This implies that there is no essential difference between the two types of experiment as far as the stability characteristics are concerned, for these depend entirely upon the parameters entering into equations (2·16) to (2·20) and upon the similar boundary conditions at $z = 0$ and $z = h$.

We consider next the boundary conditions of the problem. At the side wall of the vessel the exact boundary conditions will evidently be $u = v = w = 0$ at $r = r_0$, but since the equations themselves are not valid right up to $r = r_0$, this stringent condition will be relaxed and we shall take the condition to be the vanishing of the normal velocity at $r = r_0$, that is,

$$u = 0, \quad r = r_0. \tag{2·21}$$

This is the condition at the edge of the boundary layer, and (2·21) implies in fact that the boundary layer at the side wall is ignored. If there are two concentric cylinders then there will be a second condition similar to (2·21); but most of the work hereafter will refer to the single-sided problem. If we were able to solve the problem exactly from this point onwards the appropriate conditions at $z = 0$, the base of the liquid at $z = h$ the free upper surface of the liquid would be as follows:

$$u = v = w = 0, \quad z = 0;$$

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = w = 0, \quad z = h;$$

and these velocity conditions transform simply into

$$\xi = \eta \quad = W = 0, \quad z = 0;$$

$$\frac{\mathrm{d}\xi}{\mathrm{d}z} = \frac{\mathrm{d}\eta}{\mathrm{d}z} = W = 0, \quad z = h.$$

Likewise there will be conditions imposed at $z = 0$ and $z = h$ on the heat flow passing through these surfaces. If it is assumed that no heat flows across the upper free surface then

$$\frac{d\tau}{dz} = 0 \quad \text{at} \quad z = h,$$

and if it is assumed that the perturbation receives no heat supply from the base $z = 0$, so that it must acquire its energy from the liquid interior, then we could take

$$\frac{d\tau}{dz} = 0 \quad \text{at} \quad z = 0$$

also. However, the set of equations (2·16) to (2·20), even though they are ordinary simultaneous equations, are difficult to handle in the general case, being of the eighth order when eliminations are completely carried out. Consequently it is necessary to simplify the approach, in the first instance at least, by using physical arguments.

It is quite certain that the velocity field in its vertical variation will contain at least three important regions. Near the base of the cylinder where boundary-layer effects predominate, it is quite certain that the second-order differential terms such as $\mu d^2\xi/dz^2$, $\mu d^2\eta/dz^2$ and $\mu d^2W/dz^2$ will be of considerable importance. Likewise in a narrow region near the free upper surface these terms will be of importance but in the intermediate region, which is the greatest region in extent, it is likely that these terms play a secondary role only. It has been shown (Davies 1953) that in the symmetric problem the boundary-layer thickness is, in fact, $h/R^{\frac{1}{2}}$, where $R = \Omega\rho_0 h^2/\mu$, and in a typical case when $\Omega = 6\pi/60$ per sec., $\rho_0 = 1$, $h = 3$ cm and $\mu = 0{\cdot}01$, this gives $R = 300$ approximately and the boundary-layer thickness is about 2 mm. Accordingly, for the flow outside the narrow boundary layers we shall take the equations in the modified form given by

$$(i\rho_0\sigma' + \mu\beta^2)\,\xi + 2i\rho_0\Omega'\eta = 0, \tag{2·22}$$

$$(i\rho_0\sigma' + \mu\beta^2)\,\eta + 2i\rho_0\Omega'\xi = 2\beta(P - \gamma\chi_0), \tag{2·23}$$

$$(i\rho_0\sigma' + \mu\beta^2)\,W - g\alpha\tau = -\frac{d}{dz}(P - \gamma\chi_0), \tag{2·24}$$

$$\alpha\{i\sigma'\tau - im\Theta_H F(z)\,P + \Theta_V W\} = \tfrac{1}{2}\rho_0\beta\eta + \rho_0\frac{dW}{dz}, \tag{2·25}$$

$$\left(i\rho_0\sigma' + \frac{k\beta^2}{c_v}\right)\tau - i\rho_0 m\Theta_H F(z)\,P + \rho_0\Theta_V W = 0, \tag{2·26}$$

and we can now define the function $F(z)$ which was introduced in (1·23) to be

$$F(z) = \frac{2\rho_0\Omega'}{4\rho_0\Omega'^2 + (i\rho_0\sigma' + \mu\beta^2)^2}, \tag{2·27}$$

this being the approximate unsteady geostrophic value obtained from the pair of equations

$$(i\rho_0\sigma' + \mu\beta^2)\,u - 2\rho_0\Omega' v = 0,$$

$$(i\rho_0\sigma' + \mu\beta^2)\,v + 2\rho_0\Omega' u = -\frac{1}{r}\frac{\partial p}{d\phi},$$

in which
$$u = -\frac{1}{r}F(z)\frac{\partial p}{\partial\phi}.$$

Since in equations (2·22) to (2·26) the second differentials of velocity and temperature have been ignored the appropriate boundary conditions will be

$$\left.\begin{aligned} w = 0, \quad z = 0, \\ w = 0, \quad z = h, \end{aligned}\right\} \tag{2·28}$$

which are the inviscid conditions, and the problem is then self-consistent. The equations (2·22) to (2·26) contain the viscosity through the term $\mu\beta^2$ and the conduction through the term $k\beta^2/c_v$, and the presence of these terms will indicate the direction in which viscosity and conductivity influence the large-scale motion. The ignoring of the second derivative terms in fact is equivalent to the assumption that the viscous effect is directly proportional to the velocity and the conduction effect is directly proportional to the temperature. The viscous effect and conductivity effect is probably small in this problem, but we shall retain these modified forms of viscosity and conductivity, since they play an important part in the later discussion of the singularities. At this point, however, we shall ignore the term χ_0 on the right-hand sides of (2·23) and (2·24), since it is equal to $-\alpha k\beta^2\tau/\rho_0^2 c_v$, and the combination of $\alpha k\mu$ in the term $\gamma\chi$ makes this quite negligible. This term could well have been ignored earlier in fact, but it is of interest to have shown that the complete perturbation equations, with χ included, could be successfully tackled by the present method.

Before proceeding any further with the development of the equations (2·22) to (2·26) it is convenient to introduce non-dimensional variables $\bar{\xi}$, $\bar{\eta}$, $\overline{W}$, $\bar{\tau}$, $\overline{P}$ in place of ξ, η, W, τ and P; accordingly, we shall write

$$z = h\zeta, \quad a = \beta h, \tag{2·29}$$

so that the liquid now is contained in the range $0 \leqslant \zeta \leqslant 1$;

$$\left.\begin{aligned} \xi &= 2R_H\Omega r_0\bar{\xi}, & P &= 2\rho_0 R_H \Omega^2 r_0^2 \overline{P}, \\ \eta &= 2R_H\Omega r_0\bar{\eta}, & \chi_0 &= 2aR_H\Omega r_0\bar{\chi}/h, \\ W &= 2R_H\Omega r_0 a\overline{W}, & \tau &= \tfrac{1}{2}r_0^2\Theta_H\bar{\tau}; \end{aligned}\right\} \tag{2·30}$$

and we introduce the following non-dimensional parameters:

$$f = \frac{\sigma}{2\Omega}, \quad R = \frac{\Omega\rho_0 h^2}{\mu}, \quad K = \frac{\Omega\rho_0 h^2 c_v}{k}, \quad R_H = \frac{g\alpha h\Theta_H}{4\rho_0\Omega^2}, \quad R_v = \frac{g\alpha\Theta_V}{4\rho_0\Omega^2}. \tag{2·31}$$

In (2·31) R is a Reynolds number for the flow, K a Péclet number, R_H has already been mentioned in (1·18), R_v is a Richardson number (being of the form $g(\partial\rho/\partial z)/\rho(\partial u/\partial z)^2$), and f is a non-dimensional frequency. The equations (2·22) to (2·26) then become

$$s\bar{\xi} + \mathrm{i}(1+2R_H\zeta)\,\bar{\eta} = 0, \tag{2·32}$$

$$s\bar{\eta} + \mathrm{i}(1+2R_H\zeta)\,\bar{\xi} = \beta r_0\overline{P}, \tag{2·33}$$

$$\frac{2ah}{r_0}s\overline{W} - \tau = -\frac{\mathrm{d}P}{\mathrm{d}\zeta}, \tag{2·34}$$

$$\frac{2r_0\Omega^2}{g}\left\{\mathrm{i}(f+mR_H\zeta)\,\bar{\tau} - \frac{\mathrm{i}mR_H(1+2R_H\zeta)}{(1+2R_H\zeta)^2+s^2}\overline{P} + \frac{2ah}{r_0}R_v\overline{W}\right\} = a\left(\tfrac{1}{2}\bar{\eta} + \frac{d\overline{W}}{\mathrm{d}\zeta}\right), \tag{2·35}$$

$$s'\tau - \frac{\mathrm{i}mR_H(1+2R_H\zeta)}{(1+2R_H\zeta)^2+s^2}\overline{P} + \frac{2ah}{r_0}R_v\overline{W} = 0, \tag{2·36}$$

where, for convenience, we have written

$$s = \mathrm{i}(f+mR_H\zeta)+\frac{a^2}{2R}, \tag{2·37}$$

$$s' = \mathrm{i}(f+mR_H\zeta)+\frac{a^2}{2K}. \tag{2·38}$$

It is most convenient now to derive the differential equation for $\overline{W}$ from the set (2·32) to (2·36), since the boundary conditions are expressly imposed upon $\overline{W}$ being

$$\left.\begin{aligned}\overline{W} &= 0, \quad \zeta = 0,\\ \overline{W} &= 0, \quad \zeta = 1.\end{aligned}\right\} \tag{2·39}$$

From (2·32) and (2·33) we obtain

$$\frac{\mathrm{i}\overline{\xi}}{1+2R_H\zeta} = \frac{\overline{\eta}}{s} = \frac{\beta r_0\overline{P}}{(1+2R_H\zeta)^2+s^2}. \tag{2·40}$$

Likewise from (2·35) and (2·36) we have

$$-\frac{ar_0\Omega^2}{Kg}\overline{\tau} = \tfrac{1}{2}\overline{\eta}+\frac{\mathrm{d}\overline{W}}{\mathrm{d}\zeta}, \tag{2·41}$$

and substituting for $\overline{\eta}$ from (2·40) this gives

$$-\frac{ar_0\Omega^2}{Kg}\overline{\tau} = \frac{\frac{1}{2}\beta r_0 s}{(1+2R_H\zeta)^2+s^2}\overline{P}+\frac{\mathrm{d}\overline{W}}{\mathrm{d}\zeta}. \tag{2·42}$$

When we substitute for $\overline{\tau}$ from (2·34) in (2·42) we obtain the following relation between $\overline{P}$ and $\overline{W}$:

$$-\frac{ar_0\Omega^2}{Kg}\left\{\frac{\mathrm{d}\overline{P}}{\mathrm{d}\zeta}+\frac{2ah}{r_0}s\overline{W}\right\} = \frac{\frac{1}{2}\beta r_0 s}{(1+2R_H\zeta)^2+s^2}\overline{P}+\frac{\mathrm{d}\overline{W}}{\mathrm{d}\zeta}. \tag{2·43}$$

Similarly, by eliminating $\overline{\tau}$ between (2·36) and (2·34) we obtain a second relation between $\overline{P}$ and $\overline{W}$, namely,

$$s'\left(\frac{\mathrm{d}\overline{P}}{\mathrm{d}\zeta}+\frac{2ah}{r_0}s\overline{W}\right) = \frac{\mathrm{i}mR_H(1+2R_H\zeta)}{(1+2R_H\zeta)^2+s^2}\overline{P}-\frac{2ah}{r_0}R_v\overline{W}. \tag{2·44}$$

The elimination of $\overline{P}$ between (2·43) and (2·44) leads to the desired second-order differential equations for $\overline{W}$. In the general case we can express this equation in the form

$$s'\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{\lambda_1}{\lambda_2}\left(s'\frac{\mathrm{d}\overline{W}}{\mathrm{d}\zeta}-A\overline{W}\right)\right\}+\frac{\mathrm{i}mR_H(1+2R_H\zeta)}{\lambda_2}\left\{A\overline{W}-s'\frac{\mathrm{d}\overline{W}}{\mathrm{d}\zeta}\right\}-\frac{2ah}{r_0}(R_v+ss')\,\overline{W} = 0, \tag{2·45}$$

where

$$\left.\begin{aligned}A &= 2a^2h\Omega^2R_v/Kg,\\ \lambda_1 &= (1+2R_H\zeta)^2+s^2,\\ \lambda_2 &= \mathrm{i}mR_H\left(\frac{ar_0\Omega^2}{Kg}\right)(1+2R_H\zeta)+\tfrac{1}{2}\beta r_0 ss',\end{aligned}\right\} \tag{2·46}$$

but we may simplify this equation considerably by investigating the orders of magnitude of the various terms. In order to do this we shall use the typical values $\Omega = 0{\cdot}3$ rad/s, $f = 0{\cdot}05$, $\rho_0 = 1$, $r_0 = 15$ cm, $h = 3$ cm, $g = 10^3$, $\Theta_H = 0{\cdot}16$, $\Theta_V = 1{\cdot}45$, $R_H = 0{\cdot}3$, $R_v = 1{\cdot}0$, $R = 300$, $K = 3000$; the second quantity in this list is a typical observed value for f. In the expression for λ_2 the term $mR_H(ar_0\Omega^2/Kg)$ bears to $\frac{1}{2}\beta r_0 ss'$ the ratio 10^{-2} to 1 when s and s' are chosen to be each equal to $\mathrm{i}f$. Thus there will be no important error introduced if we write

$$\lambda_2 = \tfrac{1}{2}\beta r_0 ss'. \tag{2·47}$$

Likewise, if we compare the term $\mathrm{i}mR_H(1+2R_H\zeta)\,A\overline{W}/\lambda_2$ with $2ahR_v\overline{W}/r_0$ we find that the coefficients of $\overline{W}$ here are in the ratio 10^{-3} to 1. Similarly the term containing A in the leading expression of (2·45), namely $As'\,\mathrm{d}(\lambda_1\overline{W}/\lambda_2)/\mathrm{d}\zeta$, may be safely ignored compared with later terms in $\overline{W}$ and $\mathrm{d}\overline{W}/\mathrm{d}\zeta$. Thus we may write (2·45) in the form

$$s'\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\frac{\lambda_1}{s}\frac{\mathrm{d}\overline{W}}{\mathrm{d}\zeta}\right)-\frac{\mathrm{i}mR_H}{s}(1+2R_H\zeta)\frac{\mathrm{d}\overline{W}}{\mathrm{d}\zeta}-a^2(R_v+ss')\,\overline{W}=0, \tag{2·48}$$

and it is easily seen that the reason for this simplification of the differential equation is essentially the smallness of the factor $ar_0\Omega^2/Kg$ which appears on the left-hand side of (2·41), (2·42) and (2·43). When the term on the left-hand side of (2·41) is ignored we obtain the result $\frac{1}{2}\overline{\eta}+\mathrm{d}W/\mathrm{d}\zeta=0$; in its original context this would read $\operatorname{div}\mathbf{V}=0$, so that the above simplification is merely an expression of the physical result that the process of conductivity is negligible here in its influence upon the divergence of a fluid element. This could probably have been surmised initially, but it is more satisfactory to do so at this stage.

We now discuss equation (2·48). It may be written in the form

$$\lambda_1 ss'\frac{\mathrm{d}^2\overline{W}}{\mathrm{d}\zeta^2}+\frac{\mathrm{d}\overline{W}}{\mathrm{d}\zeta}\left\{-\mathrm{i}mR_Hs(1+2R_H\zeta)+ss'\frac{\mathrm{d}\lambda_1}{\mathrm{d}\zeta}-s'\lambda_1\frac{\mathrm{d}s}{\mathrm{d}\zeta}\right\}-a^2s^2(R_v+ss')\,\overline{W}=0, \tag{2·49}$$

and it is clear from this form that the equation has singularities at the points where s, s' and λ_1 vanish. From the definition of s in (2·37) we see that s vanishes at $\zeta=\zeta_1$, where

$$\zeta_1=-\frac{f}{mR_H}+\frac{\mathrm{i}a^2}{2RmR_H}; \tag{2·50}$$

similarly, from (2·38), s' vanishes at $\zeta=\zeta_2$, where

$$\zeta_2=-\frac{f}{mR_H}+\frac{\mathrm{i}a^2}{2KmR_H}; \tag{2·51}$$

and, from (2·46), λ_1 vanishes at $\zeta=\zeta_3$ and $\zeta=\zeta_4$, where

$$\zeta_3R_H(m+2)=-1-f+\frac{\mathrm{i}a^2}{2R}, \tag{2·52}$$

$$\zeta_4R_H(m-2)=1-f+\frac{\mathrm{i}a^2}{2R}. \tag{2·53}$$

The quantity ζ in the present problem is real and varies between 0 and 1, and thus all the singularities lie at points in a complex ζ plane which are not on the real axis if a^2 is a real quantity. It is interesting to note, however, that if $R\to\infty$ and $K\to\infty$ then there will certainly be at least one singularity in the real range $0\leqslant\zeta\leqslant1$ when m is sufficiently large, and the presence of such a singularity gives rise to peculiar difficulties in satisfying the boundary conditions (2·39). Thus the retention of viscosity and conductivity, though in a form which is not exact, serves a most useful purpose in the problem. The remaining sections of this paper will discuss various aspects of equation (2·48).

3. The barotropic solution

In the expressions for σ' and Ω' given in (2·8) and (2·9) it has been observed that an interpretation of their variation is that the effective frequency and effective angular velocity depend upon z. A very much simplified theory will evidently result if we assume that the effective frequency and effective angular velocity show no variation with height. This is an assumption made frequently in the corresponding meteorological problems. The immediate advantage of this barotropic assumption is that there is no possibility of any singularity arising in the differential equation (2·48), and we shall investigate this case in detail when R and K are sufficiently large to ignore the terms $a^2/2R$ and $a^2/2K$. The case in which $a^2/2R$ and $a^2/2K$ are retained will be discussed briefly at the end of this section.

In equation (2·48) we now replace $s = if$, $s' = if$, $\lambda_1 = 1-f^2$, and it then becomes

$$(1-f^2)\frac{d^2\overline{W}}{d\zeta^2}-\frac{mR_H}{f}\frac{d\overline{W}}{d\zeta}-a^2(R_v-f^2)\,\overline{W}=0. \tag{3·1}$$

The condition to be satisfied by the coefficients of an equation of the form

$$a_1\frac{d^2\overline{W}}{d\zeta^2}+b_1\frac{d\overline{W}}{d\zeta}+c_1\overline{W}=0 \tag{3·2}$$

when $\overline{W}=0$ at $\zeta=0$ and $\zeta=1$ is

$$\frac{b_1^2-4a_1c_1}{a_1^2}=-4\pi^2n^2 \quad (n=1,2,3,\ldots). \tag{3·3}$$

Applying this result to (3·1) we obtain

$$\frac{m^2R_H^2}{f^2}+4a^2(1-f^2)\,(R_v-f^2)=-4\pi^2n^2(1-f^2)^2, \tag{3·4}$$

and this, of course, is a frequency equation, giving f when R_H, R_v and a are known. Interest centres principally upon small values of f, since with the long-wave patterns observed in the experiment the angular velocity relative to the rotating vessel, namely, $-2\Omega f/m$, is known to be small. Accordingly we can write (3·4) in the simplified form

$$\frac{m^2R_H^2}{f^2}=-4(a^2R_v+\pi^2n^2), \tag{3·5}$$

since R_v is of order unity. In this relation the quantity a is undefined, and in order to obtain a second relation between f, a, etc., we use the boundary condition at $r=r_0$ given in (2·21). Using (2·22), which in the present case may be written in the form

$$f\xi+\eta=0, \tag{3·6}$$

together with (2·9), we have

$$2u=\xi(z)\left\{\frac{mC_m(\beta r)}{\beta r}+fC'_m(\beta r)\right\}. \tag{3·7}$$

Thus the boundary condition will be satisfied in the single-sided problem by choosing $C_m=J_m$ and by making β satisfy the equation

$$\frac{mJ_m(\beta r_0)}{\beta r_0}+fJ'_m(\beta r_0)=0; \tag{3·8}$$

the annulus problem will not be considered here. Since $a=\beta h$, the parameter β can be eliminated between (3·5) and (3·8) to provide a relation between f, r_0, h, R_v and R_H which is the true frequency equation of the problem. It will be observed that if β is real and, accordingly, a real, then (3·5) will lead to a purely imaginary value for f so that there is no neutral progressive wave. On the other hand, when β is purely imaginary it is easily shown that progressive wave solutions are possible. If we solve for β from (3·5) we obtain

$$\beta^2h^2 = -\left\{\pi^2n^2+\frac{m^2R_H^2}{4f^2}\right\}\Big/R_v, \tag{3·9}$$

and thus

$$\beta r_0 = \mathrm{i}\lambda, \tag{3·10}$$

where

$$\lambda^2 = \left(\pi^2n^2+\frac{m^2R_H^2}{4f^2}\right)\frac{r_0^2}{h^2R_v}. \tag{3·11}$$

Substituting for β in (3·8) we have

$$\frac{mJ_m(\mathrm{i}\lambda)}{\mathrm{i}\lambda}+fJ_m'(\mathrm{i}\lambda) = 0,$$

where λ is real. Using the modified Bessel function I_m which is defined by the relation $I_m(z) = \mathrm{i}^{-m}J_m(\mathrm{i}z)$ (Whittaker & Watson, *Modern analysis*, p. 372), this relation can be written in the form

$$\frac{mI_m(\lambda)}{\lambda}+fI_m'(\lambda) = 0, \tag{3·12}$$

and for the particular barotropic flow which we have considered in this section this relation is exact. Equation (3·12) may be simplified as follows in the present case. For small values of f, λ is a large quantity, and it is permissible to use the asymptotic formulae for $I_m(\lambda)$ and $I_m'(\lambda)$, namely,

$$I_m(\lambda) \sim \frac{\mathrm{e}^\lambda}{(2\pi\lambda)^{\frac{1}{2}}}\left\{1-\frac{(4m^2-1)}{8\lambda}+o\left(\frac{1}{\lambda^2}\right)\right\},$$

$$I_m'(\lambda) \sim \frac{\mathrm{e}^\lambda}{(2\pi\lambda)^{\frac{1}{2}}}\left\{1-\frac{(4m^2+3)}{8\lambda}+o\left(\frac{1}{\lambda^2}\right)\right\}.$$

Accordingly (3·12) becomes approximately

$$\frac{m}{\lambda}\left\{1-\frac{4m^2-1}{\lambda}\right\}+f\left\{1-\frac{4m^2+3}{8\lambda}\right\} = 0,$$

and f is therefore given by

$$f = -\frac{m}{\lambda}\left\{1+\frac{1}{2\lambda}+o\left(\frac{1}{\lambda^2}\right)\right\}. \tag{3·13}$$

If only the first term is taken on the right-hand side of (3·13), so that $f = -m/\lambda$, it follows that the frequency equation is

$$f^2\left\{\pi^2n^2+\frac{m^2R_H^2}{4f^2}\right\}\frac{r_0^2}{h^2R_v} = m^2 \tag{3·14}$$

when we use (3·11), and the solution for f is given by

$$n^2\pi^2f^2 = m^2\left\{\frac{h^2}{r_0^2}R_v-\tfrac{1}{4}R_H^2\right\}. \tag{3·15}$$

This frequency equation will be valid provided the quantity in the brackets is sufficiently small. Evidently the condition for stability of the wave is

$$\frac{h^2}{r_0^2}R_v>\tfrac{1}{4}R_H^2, \tag{3·16}$$

since when this is satisfied a real value of f results. The quantity n in (3·15) is an integer, and in the most important case when W has no nodal surfaces in the region $0<\zeta<1$, which is the case usually observed in experiments, we can take $n = 1$. With $n = 1$ and (3·16) satisfied the formula for the angular velocity ω of propagation of the wave system becomes

$$\frac{\omega}{\Omega}=\frac{2f}{m}$$
$$=\pm\frac{2}{\pi}\left\{\frac{h^2}{r_0^2}R_v-\tfrac{1}{4}R_H^2\right\}^{\frac{1}{2}}, \tag{3·17}$$

the upper sign refers to a wave propagated in the E→W direction and the lower sign to one propagated in the W→E direction. The quantities R_v and R_H are independent of m, and thus the formula (3·17) is a non-dispersive propagation formula, which agrees with a statement due to Hide. Apart from this very minor qualitative agreement it is most difficult to compare (3·17) with any of Hide's empirical formulae, since he does not introduce the parameter R_v explicitly in his results. In the present theory R_H and R_v are introduced as two independent parameters, but in order to reproduce Hide's empirical formulae it seems to be necessary to postulate some relation between them. It is more useful at this stage of the theory, however, to investigate (3·16) in greater detail, since the result is quite decisive.

If we substitute for R_v and R_H in (3·16) it may be rearranged in the form

$$\Omega^2>g\alpha r_0^2\,\Theta_H^2/16\rho_0\,\Theta_V \tag{3·18}$$

for stability of the wave pattern, and we may state that according to barotropic theory no neutral progressive stable wave exists when

$$\Omega^2<g\alpha r_0^2\,\Theta_H^2/16\rho_0\,\Theta_V. \tag{3·19}$$

When this inequality (3·19) is satisfied, (3·15) does not indicate whether the corresponding wave is exponentially damped or exponentially increasing in amplitude, but it is likely that $\Omega = \Omega_{\text{crit.}}$, where

$$\Omega_{\text{crit.}}^2 = g\alpha r_0^2\,\Theta_H^2/16\rho_0\,\Theta_V \tag{3·20}$$

represents an important and critical stage in the value of this parameter. It we take typical experimental values $\rho_0 = 1$, $r_0 = 15$ cm, $\alpha = 2{\cdot}5\times10^{-4}$, $g = 10^3$, $\Theta_H = 0{\cdot}16$, $\Theta_V = 1{\cdot}45$, we obtain $\Omega_{\text{crit.}} = 0{\cdot}25$, and this gives as the critical number of revolutions per minute $(= 60\Omega_{\text{crit.}}/2\pi)$ the value 2·4. The observed critical number of revolutions when spiral flow gives way to wave flow is a number near 3 for the above data, and, apparently, barotropic theory gives a quantitative result of the correct order of magnitude. Further agreement with experiment is certainly not forthcoming however, for, as stated in the introduction the theory must also explain why, for example, a three-wave pattern changes to a four-wave pattern and there is nothing in (3·18) to explain such a feature. In order to see how the above theory differs from Hide's empirical formulae it is useful at this stage to summarize his principal results.

Hide obtains a formula for the wave velocity in the form

$$R_0 = -(0{\cdot}0288\pm0{\cdot}0008)\,\Theta\,\frac{(b-a)}{\frac{1}{2}(b+a)}, \tag{3·21}$$

where R_0, the Rossby number, is defined to be the ratio of the mean relative zonal angular speed of flow (this is identified by Hide with the angular velocity of propagation of the wave pattern) to the angular velocity of the cylinder, b is the outer cylinder radius, a the inner cylinder radius and Θ is defined by

$$\Theta = \frac{gh}{\frac{1}{2}\Omega^2(b^2-a^2)}\left|\frac{\Delta\rho}{\rho}\right|\left(\frac{1}{2}\frac{b+a}{b-a}\right), \tag{3·22}$$

where $\Delta\rho$ is the difference in density between the liquid at $r = a$ and $r = b$. We may, for the purpose of comparison, take $a = 0$, $b = r_0$ and $|\Delta\rho/\rho| = \alpha(\Delta T)/\rho$, and hence, in our notation,

$$\Theta = \frac{g\alpha h\Theta_H}{2\rho_0\Omega^2} = 2R_H. \tag{3·23}$$

Thus, assuming R_0 to be given by ω/Ω, we have, in our notation,

$$\frac{\omega}{\Omega} = -(0·0288 \pm 0·0008)\, 4R_H = -0·13R_H. \tag{3·24}$$

This experimental result must be compared with (3·17), and, as has been stated earlier, no correspondence between the results is possible unless h^2R_v/r_0^2 is directly proportional to R_H^2. Hide states also that spiral flow changes to wave flow when $\Theta = \Theta_{\text{crit.}} = 1·58$; this corresponds to $R_H = 0·79$, and again no comparison is possible with (3·16).

It seems therefore that a barotropic type of solution is probably not adequate in explaining the principal features of the experiment, and a more elaborate theory will be established which will incorporate baroclinic features.

When viscosity and conductivity are retained and the simplifying assumption $R = K$ is made, the barotropic theory as expounded here is modified only in that $f - \mathrm{i}a^2/2R$ replaces f everywhere in the analysis. It follows that the equation corresponding to (3·15) will be

$$\pi^2\left(f - \frac{\mathrm{i}a^2}{2R}\right)^2 = m^2\left\{\frac{h^2}{r_0^2}R_v - \tfrac{1}{4}R_H^2\right\}.$$

The solutions for f are in general complex and no neutral waves can exist. Since $a^2 < 0$ the wave forms will in general be unstable.

4. The modified baroclinic solution

In this section it is proposed to obtain a solution of the equation (2·49) in which the singularities ζ_1 and ζ_2 of (2·50) and (2·51) are retained, ζ_3 and ζ_4 of (2·52) and (2·53) are ignored, and the conductivity and viscosity are made equal to zero. When R and K tend to infinity it will be noted that all the singularities ζ_s $(s = 1, 2, 3, 4)$ lie on the real axis, that in particular $\zeta_1 = \zeta_2 = -f/mR_H$, and these two will lie within the range $0 < \zeta < 1$ if f is negative and $|f| < mR_H$; this case is the important one, since we are concerned principally with progressive waves which are moving slowly in the west-east direction (i.e. in the same direction as the rotation). It is evident, therefore, that in any adequate theory of progressive east-west waves the singularities ζ_1 and ζ_2 must be taken into account in the solution of (2·49). The singularity ζ_3 always lies outside the range $0 < \zeta < 1$ even when R is infinite since $|f| \ll 1$, so it would seem that ζ_3 is not as important a singularity as ζ_1 or ζ_2. The singularity ζ_4, on the other hand, will lie in the range $0 < \zeta < 1$ when R is infinite, $|f| \ll 1$ and

$R_H(m-2)>1$. Taking R_H to be of magnitude 0·3 it follows that m must attain the value 6 before the singularity ζ_4 can lie in $0<\zeta<1$. Even though ζ_4 may be of importance for sufficiently large m we shall disregard it in the present section and will try to assess its influence in the next section. Accordingly, we shall make the following simplifying assumptions in equation (2·49):

(i) in the coefficient of $d^2\overline{W}/d\zeta^2$ the λ_1 term will be replaced by unity, which is its effective value at $\zeta=0$,

(ii) the coefficient of $d\overline{W}/d\zeta$ is of the form

$$s\,imR_H\{-2-2R_H\zeta+R_H^2\zeta^2(4-m^2)+o(f)\},$$

and it will be assumed that this term takes the value $-2s\,imR_H$,

(iii) the coefficient of $\overline{W}$ will be taken in the form $-a^2s^2R_v$.

These three simplifying assumptions look after the singularity $s=0$ effectively, and in all three cases it is assumed that the remaining terms take their surface values. Equation (2·49) then becomes

$$s^2\frac{d^2\overline{W}}{d\zeta^2}-2s\,imR_H\frac{d\overline{W}}{d\zeta}-a^2s^2R_v\overline{W}=0, \tag{4·1}$$

where $s=i(f+mR_H\zeta)$, and we shall assume that f is a small negative quantity such that $-1<f/mR_H<0$.

It we take a new origin at the singular point and write

$$\zeta^*=\zeta+\frac{f}{mR_H}=\zeta-\zeta_0, \tag{4·2}$$

equation (4·1) becomes

$$\frac{d^2\overline{W}}{d\zeta^{*2}}-\frac{2}{\zeta^*}\frac{d\overline{W}}{d\zeta^*}-a^2R_v\overline{W}=0, \tag{4·3}$$

and it is easily shown that the complete solution of (4·3) is

$$\overline{W}=\zeta^{*\frac{3}{2}}\{AJ_{\frac{3}{2}}(i\zeta^*aR_v^{\frac{1}{2}})+BJ_{-\frac{3}{2}}(i\zeta aR_v^{\frac{1}{2}})\}, \tag{4·4}$$

where A, B are two arbitrary constants and where $J_{\pm\frac{3}{2}}(\eta^*)$ are Bessel functions which are expressible in the closed form

$$J_{\frac{3}{2}}(\eta^*)=\left(\frac{2\eta^*}{\pi}\right)^{\frac{1}{2}}\left\{\frac{\sin\eta^*}{\eta^{*2}}-\frac{\cos\eta^*}{\eta^*}\right\},$$

$$J_{-\frac{3}{2}}(\eta^*)=-\left(\frac{2\eta^*}{\pi}\right)^{\frac{1}{2}}\left\{\frac{\sin\eta^*}{\eta^*}+\frac{\cos\eta^*}{\eta^{*2}}\right\}.$$

It will be noted that $J_{\frac{3}{2}}(\eta^*)$ tends to zero as $\eta^*\to 0$ and $J_{-\frac{3}{2}}(\eta^*)\to\infty$ as $\eta^*\to 0$, but the solution for $\overline{W}$ has no singularity at $\zeta^*=0$, since the $\zeta^{*\frac{3}{2}}$ in (4·4) disposes of the singularity in $J_{-\frac{3}{2}}$ at $\zeta^*=0$. Thus with a slight adjustment of the constants we can take

$$\overline{W}=A(\sin\eta^*-\eta^*\cos\eta^*)+B(\eta^*\sin\eta^*+\cos\eta^*), \tag{4·5}$$

where $\eta^*=i\zeta^*aR_v^{\frac{1}{2}}$. The boundary conditions upon $\overline{W}$ are $\overline{W}=0$ at $\zeta=0$ and $\zeta=1$, thus if we write

$$\eta_0=\frac{ifaR_v^{\frac{1}{2}}}{mR_H},\quad \eta_1=i\left(1+\frac{f}{mR_H}\right)aR_v^{\frac{1}{2}}, \tag{4·6}$$

then

$$\left.\begin{aligned}A(\sin\eta_0-\eta_0\cos\eta_0)+B(\eta_0\sin\eta_0+\cos\eta_0)&=0,\\ A(\sin\eta_1-\eta_1\cos\eta_1)+B(\eta_1\sin\eta_1+\cos\eta_1)&=0.\end{aligned}\right\} \tag{4·7}$$

The consistency equation of the pair of equations (4·7) reduces to

$$(1+\eta_0\eta_1)\sin(\eta_0-\eta_1)-(\eta_0-\eta_1)\cos(\eta_0-\eta_1) = 0, \tag{4·8}$$

and using (4·6) this becomes

$$\left\{1-\frac{fa^2R_v}{mR_H}\left(1+\frac{f}{mR_H}\right)\right\}\sin(\mathrm{i}\,aR_v^{\frac{1}{2}})-\mathrm{i}\,aR_v^{\frac{1}{2}}\cos(\mathrm{i}\,aR_v^{\frac{1}{2}}) = 0. \tag{4·9}$$

This represents the first relation between f, β, R_v and R_H. A second relation between these parameters is obtained by satisfying $u = 0$ at $r = r_0$. This cannot be done exactly in the present case, since σ' and Ω' are now functions of z. If we again consider the case in which liquid occupies the whole region $0 \leqslant r \leqslant r_0$, then the appropriate solution for u from (2·9) and (2·22), with $\mu = 0$, is

$$2u = \left\{\frac{mJ_m(\beta r)}{\beta r}+\frac{\sigma'}{2\Omega'}J'_m(\beta r)\right\}\xi(z), \tag{4·10}$$

where $\sigma' = 2\Omega(f+mR_H\zeta)$ and $\Omega' = \Omega(1+2R_H\zeta)$. If we disregard the ζ terms in the expressions of σ' and Ω' the second relation between the parameters is

$$\frac{mJ_m(\beta r_0)}{\beta r_0}+fJ'_m(\beta r_0) = 0. \tag{4·11}$$

We first of all investigate the possibility of a solution of (4·9) and (4·11) in which f is purely real and small and β is a large imaginary value. Suppose we introduce λ as in (3·10), then with λ large (4·11) leads to

$$f+\frac{m}{\lambda} = 0, \tag{4·12}$$

as in (3·13). Thus, since $a = \beta h = \mathrm{i}\lambda h/r_0$, (4·9) becomes

$$\left\{1+\frac{f\lambda^2h^2R_v}{mR_Hr_0^2}\left(1+\frac{f}{mR_H}\right)\right\}\sin\left(\frac{\lambda h R_v^{\frac{1}{2}}}{r_0}\right)-\frac{\lambda h R_v^{\frac{1}{2}}}{r_0}\cos\left(\frac{\lambda h R_v^{\frac{1}{2}}}{r_0}\right) = 0, \tag{4·13}$$

and if we combine (4·12) and (4·13) and write, for simplicity,

$$E = \frac{hR_v^{\frac{1}{2}}}{r_0R_H}, \quad x = \lambda R_H E, \tag{4·14}$$

then x has to be determined from

$$(1+E^2-xE)\sin x - x\cos x = 0$$

or

$$\cot x = \frac{1+E^2}{x}-E. \tag{4·15}$$

It is evident that there are an infinite set of eigenvalues x_n which are the intersections of the two curves $y = \cot x$ and $y = (1+E^2)/x-E$. As in the previous section interest centres principally upon the first one. The curve $y = (1+E^2)/x-E$ crosses the axis of x at $x = E+E^{-1} > 2$, and the first intersection of the two curves will therefore be near $x = 2\pi$ so that $x_1 = 2\pi$, $x_n = n\pi$ $(n \geqslant 2)$. Thus the first eigenvalue of λ is $\lambda_1 = 2\pi/ER_H$, and thus the principal value of f is given by

$$f = -mER_H/2\pi; \tag{4·16}$$

the angular velocity ω of propagation of the wave system is given by

$$\frac{\omega}{\Omega} = \frac{2f}{m} = -\frac{R_H E}{\pi} = -\frac{hR_v^{\frac{1}{2}}}{\pi r_0}. \tag{4·17}$$

It was noted earlier that the boundary condition $u = 0$ at $r = r_0$ could not be exactly satisfied in this case, and it is of importance to determine how sensitive (4·17) is to this condition. Using (4·10) and satisfying $u = 0$ at $r = r_0$ at the height $\zeta = \frac{1}{2}$ leads to the relation

$$\frac{mJ_m(\beta r_0)}{\beta r_0} + \frac{f + \frac{1}{2}mR_H}{1+R_H} J'_m(\beta r_0) = 0$$

in place of (4·11). Thus the relation

$$\frac{f+\frac{1}{2}mR_H}{1+R_H} + \frac{m}{\lambda} = 0$$

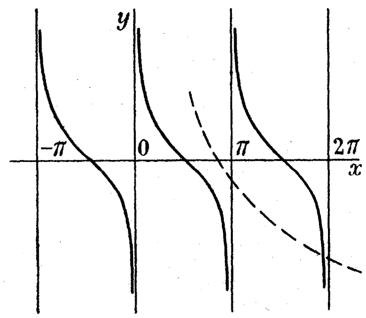

FIGURE 1. Graphs of $y = \cot x$, $y = (1+E^2)/x - E$.

replaces (4·12). Elimination of f between this equation and (4·13) leads to the relation

$$\cot x = \frac{1+E^2(1+R_H^2)}{x} - \tfrac{1}{4}x$$

in place of (4·15). The curve $y = \{1+E^2(1+R_H^2)\}/x - \frac{1}{4}x$ crosses the axis of x at

$$x = 2\{1+E^2(1+R_H^2)\}^{\frac{1}{2}},$$

and for the typical values of E and R_H in this problem this point is near the crossing point of the former curve. Thus the position of the first eigenvalue x_1 is little altered and x_1 is therefore near 2π, in fact, nearer 2π than the previous case. Thus the formulae (4·16) and (4·17) are not particularly sensitive to the boundary condition $u = 0$ at $r = r_0$.

So far we have demonstrated the existence of an infinite set of eigenvalues for f which correspond to purely imaginary values of β. The question now arises whether any other real eigenvalues of f exist or not. We deal now with the case in which f is real and β is real, and we shall assume that the second relation between f and β is (4·11). If f is small we can derive the analytical solution of (4·11) as follows. Let x_{sm} be the zeros of $J_m(x) = 0$ and assume that the solutions of

$$\frac{mJ_m(x)}{x} + fJ'_m(x) = 0†$$

for small f are given by

$$x = x_{sm} + fy_{sm} + o(f^2),$$

† If we satisfy $u=0$, $r=r_0$ at a height $\zeta=\frac{1}{2}$, the same type of method may be applied, yielding a similar formulae for βr_0 as in (4·18) but with more complicated coefficients.

where y_{sm} is independent of f. Then by applying Taylor's theorem we get simply $y_{sm} = -x_{sm}/m$; thus the solutions of (4·11) are given by

$$\beta r_0 = x_{sm}\left(1 - \frac{f}{m}\right), \tag{4·18}$$

ignoring terms of order f^2, where x_{sm} is a zero of $J_m(x) = 0$. Interest centres principally upon the first zero x_{1m} of $J_m(x) = 0$ and for future reference these are given in table 2; we make use later of the approximate result

$$x_{1m} = (\tfrac{3}{4} + \tfrac{1}{2}m)\,\pi.$$

Table 2

m	0	1	2	3	4	5	6
x_{1m}	2·4	3·8	5·1	6·4	7·6	8·8	9·9

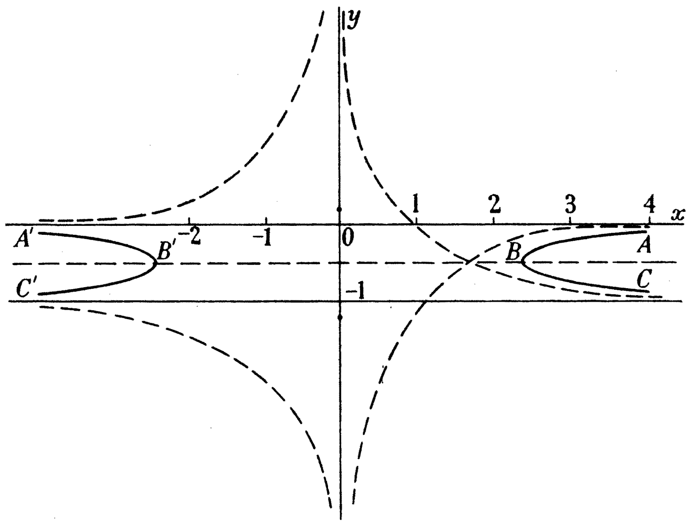

Figure 2

In the case of the first relation (4·9) we can now write

$$\left\{1 - \frac{fa^2R_v}{mR_H}\left(1 + \frac{f}{mR_H}\right)\right\} = aR_v^{\frac{1}{2}}\coth\,(aR_v^{\frac{1}{2}}). \tag{4·19}$$

If we write

$$y = f/mR_H, \quad x = aR_v^{\frac{1}{2}}, \tag{4·20}$$

equations (4·19) and (4·18) may be written more conveniently in the forms

$$1 - yx^2(1+y) = x\coth x \tag{4·21}$$

and

$$x = \frac{hR_v^{\frac{1}{2}}x_{sm}}{r_0}(1 - yR_H). \tag{4·22}$$

If the straight line (4·22) intersects the curve (4·21) a real eigen solution is possible in which both β and f are real. The curve defined by (4·21) is given in figure 2. It consists of two isolated points at (0, 0·26) and (0, −1·26), which are of no significance in the present problem, together with the two branches ABC, $A'B'C'$ which are asymptotic to $xy+1 = 0$ and $x(y+1) = 1$. The point B is (2·40, 0·5). The straight line (4·22) makes positive intercepts on the two axes, and it is evident that when the coefficients of x and y in this linear equation satisfy a particular relation the line will touch the branch ABC somewhere between B and C. The determination of this relation is of crucial importance for the stability problem; in general, this determination is quite a complicated procedure, but much of the difficulty can be avoided by using an approximate form of the curve (4·21) which is valid near the point B. If we take the origin of the curve ABC at B using the transformation $x = X+2{\cdot}40$, $y = Y-0{\cdot}5$, it is easily shown that the equation of the curve ABC in the neighbourhood of B is

$$Y^2 = a_1 X - b_1 X^2, \tag{4·23}$$

where $a_1 = 0{\cdot}0458$, $b_1 = 0{\cdot}0031$. The condition that $l_1 X + m_1 Y = n_1$ is tangent to this conic is

$$\tfrac{1}{4}a_1^2 m_1^2 = b_1 n_1^2 - a_1 l_1 n_1, \tag{4·24}$$

and in the present case the constants l_1, m_1 and n_1 are

$$\left.\begin{aligned} l_1 &= \frac{r_0}{hR_v^{\frac{1}{2}} x_{sm}}, \\ m_1 &= R_H, \\ n_1 &= 1+\tfrac{1}{2}R_H - \frac{2{\cdot}4 r_0}{hR_v^{\frac{1}{2}} x_{sm}}. \end{aligned}\right\} \tag{4·25}$$

It is easily shown that the tangent to the curve (4·23) which passes through the original O has an abscissa $X = 0{\cdot}3$ for the point of contact, and for the purposes of the present problem it is unnecessary to use the term $b_1 X^2$ in (4·23). Thus (4·24) can be simplified still further to

$$\tfrac{1}{4}a_1 m_1^2 + l_1 n_1 = 0, \tag{4·26}$$

and thus the desired criterion for contact of (4·21) and (4·22) is

$$0{\cdot}0115\frac{h^2 R_v R_H^2 x_{sm}^2}{r_0^2} + \frac{hR_v^{\frac{1}{2}} x_{sm}}{r_0}(1+\tfrac{1}{2}R_H) - 2{\cdot}4 = 0. \tag{4·27}$$

The first term in the expression on the left-hand side is, in applications, considerably smaller than the others and can usually be ignored. As a general check upon this criterion it may be observed that when $R_H \to 0$ in (4·22) that the line becomes $x = hR_v^{\frac{1}{2}} x_{sm}/r_0$, and the condition for contact is then $hR_v^{\frac{1}{2}} x_{sm}/r_0 = 2{\cdot}4$. It follows from the above that no intersection of (4·22) and (4·21) will take place if

$$0{\cdot}0115\frac{h^2 R_v R_H^2 x_{sm}^2}{r_0^2} + \frac{hR_v^{\frac{1}{2}} x_{sm}}{r_0}(1+\tfrac{1}{2}R_H) - 2{\cdot}4 < 0, \tag{4·28}$$

and this must be interpreted as the instability criterion for baroclinic flow, since, when the expression on the left-hand side of (4·28) is positive, real stable waves exist, and when negative the waves of this species have a complex value of f (see (4·33)). It is usually sufficient to deal with the instability criterion in the approximate form

$$\frac{hR_v^{\frac{1}{2}} x_{sm}}{r_0}(1+\tfrac{1}{2}R_H) < 2{\cdot}4. \tag{4·29}$$

This criterion differs very considerably from the barotropic criterion, namely,

$$\frac{hR_v^{\frac{1}{2}}}{r_0} < \tfrac{1}{2}R_H, \tag{4·30}$$

not only in the arrangement of its parameters h, r_0, R_v and R_H but also in an entirely new feature, namely, its dependence upon wave number through x_{sm}.

If we replace R_H and R_v in (4·29) by their original expressions in terms of Θ_H, Θ_V and Ω, etc., the instability criterion may be written in the form

$$\frac{h}{r_0} x_{sm}\left\{1+\frac{g\alpha h\Theta_H}{8\rho_0\Omega^2}\right\} < 2{\cdot}4\left(\frac{4\rho_0}{g\alpha\Theta_V}\right)^{\frac{1}{2}}\Omega, \tag{4·31}$$

and in this formula interest centres principally upon x_{1m}, which represents the major mode of the horizontal wave. It will be noted from the table of values of x_{1m} given earlier that x_{1m} increases steadily with m and, from (4·31), it will be noted that Ω increases with m. Alternatively, we can state that wave number $m = 0$ becomes unstable at $\Omega = \Omega_0$, wave number $m = 1$ becomes unstable at $\Omega = \Omega_1$, where $\Omega_1 > \Omega_0$, wave number $m = 2$ at $\Omega = \Omega_2 > \Omega_1 > \Omega_0$, and so on. This is similar to the behaviour in the Fultz experiment as described in the introduction to this paper. It may be noted also from (4·31) that if the

TABLE 3

Ω/Ω_c	$\frac{5}{3}$	$\frac{3}{2}$	$\frac{4}{3}$	1	$\frac{2}{3}$	$\frac{1}{2}$
$\dfrac{12(\Omega/\Omega_c)}{1+0{\cdot}15(\Omega_c/\Omega)^2}$	18·6	17·5	14·5	10	6	3·6

heating constants $h\Theta_H$ and Θ_V can be maintained constant and Ω and r_0 are also kept constant, then, for variable h, stability will result if hx_{1m} is kept constant. Thus if h increases, x_{1m} must decrease, and hence m must decrease. In other words, if the depth alone is increased, smaller and smaller wave numbers can possibly appear. The demonstration of this particular feature has been given to the author by Hide, and this appears to strengthen the claim of the above baroclinic instability formula. It should be mentioned, however, that the relation hx_{1m} = constant used above is likely to be inexact, since Θ_V and $h\Theta_H$ are not under rigid control. In view of this qualitative agreement it is worth while to investigate (4·31) quantitatively, and if we use the typical values $h = 3$ cm, $r_0 = 15$ cm, $\Theta_H = 0{\cdot}16$, $\Theta_V = 1{\cdot}45$, $\rho_0 = 1$, $g = 10^3$, $\alpha = 2{\cdot}5\times10^{-4}$ and $\Omega = \Omega_c = 0{\cdot}3\,\mathrm{s}^{-1}$ (3 rev/min, approximately) we can write (4·31) in the form

$$\tfrac{1}{5}x_{1m}\left\{1+0{\cdot}15\left(\frac{\Omega_c}{\Omega}\right)^2\right\} < 2{\cdot}4\left(\frac{\Omega}{\Omega_c}\right). \tag{4·32}$$

From table 3 we deduce that $m = 0$ becomes unstable at approximately 1·3 rev/min, $m = 1$ becomes unstable at 1·5 rev/min, $m = 2$ at 1·9 rev/min, $m = 3$ at 2·1 rev/min, $m = 4$ at 2.4 rev/min, $m = 5$ at 2·6 rev/min, and so on. The experimental range is considerably wider than this, and thus the quantitative agreement is only fair.

We now investigate the formula for the wave velocity in the present case assuming that the wave exists and is stable. Suppose we use (4·23) in the approximate form

$$Y^2 = a_1 X,$$

and we take the straight line to be $l_1 X + m_1 Y = n_1$, where l_1, m_1, n_1 are defined in (4·25). We then have

$$l_1 Y^2 + a_1 m_1 Y - a_1 n_1 = 0,$$

and solving this equation for Y ($= 0{\cdot}5 + f/mR_H$) we get the wave-velocity formula

$$\frac{f}{mR_H} + 0{\cdot}5 = \frac{1}{2l_1}\{-a_1 m_1 \pm (a_1^2 m_1^2 + 4a_1 l_1 n_1)^{\frac{1}{2}}\}.$$

We have seen that the term $a_1^2 m_1^2$ is small compared with $4a_1 l_1 n_1$, and a sufficiently accurate formula is given by

$$\frac{f}{mR_H} + 0{\cdot}5 = -\tfrac{1}{2}a_1 \frac{m_1}{l_1} \pm a_1^{\frac{1}{2}}\left(\frac{n_1}{l_1}\right)^{\frac{1}{2}},$$

or, written out in full,

$$\frac{f}{mR_H} + 0{\cdot}5 = -0{\cdot}0229\frac{h}{r_0} R_v^{\frac{1}{2}} R_H x_{sm} \pm 0{\cdot}2140\left\{\frac{(1+\frac{1}{2}R_H)\, hR_v^{\frac{1}{2}} x_{sm}}{r_0} - 2{\cdot}4\right\}^{\frac{1}{2}}. \tag{4·33}$$

It is not possible at this stage to discuss this formula in its relation to the experiment, since the parameter R_v is absent in Hide's work, but we shall return to this point in the next section.

The existence of two types of wave forms has been demonstrated, the first type possessing the wave-velocity formula (4·16), the second type possessing the wave-velocity formula (4·33). In the waves of the first type there is no suggestion of instability and this seems to be a feature of the second type only. It is possible that other wave types exist corresponding to complex values of β, but there is every likelihood that such wave types would always produce complex values of f, so that permanent waves of this species are unlikely.

The derivation of the velocity, pressure and temperature fields in the baroclinic case will be done only for the second type wave whose wave-velocity formula is (4·33). Using (4·5), (4·6) and (4·7) it follows that $\overline{W}$ can be expressed in the form

$$\overline{W} = \frac{A}{2R_H \Omega r_0 a}\left\{\left[1 - \frac{fa^2 R_v}{mR_H}(\zeta - \zeta_0)\right]\sinh(\zeta a R_v^{\frac{1}{2}}) - \zeta a R_v^{\frac{1}{2}} \cosh(\zeta a R_v^{\frac{1}{2}})\right\}, \tag{4·34}$$

and if A is treated as a real constant the solution for w can be taken in the form

$$w = A\left\{\left[1 - \frac{fa^2 R_v}{mR_H}(\zeta - \zeta_0)\right]\sinh(\zeta a R_v^{\frac{1}{2}}) - \zeta a R_v^{\frac{1}{2}} \cosh(\zeta a R_v^{\frac{1}{2}})\right\} J_m(\beta r)\cos(m\phi + \sigma t). \tag{4·35}$$

Using the relation $\eta = -\dfrac{2}{a}\dfrac{\mathrm{d}W}{\mathrm{d}\zeta}$ it then follows that

$$\eta = 2AaR_v(\zeta - \zeta_0)\left\{\frac{faR_v^{\frac{1}{2}}}{mR_H}\cosh(\zeta a R_v^{\frac{1}{2}}) + \sinh(\zeta a R_v^{\frac{1}{2}})\right\}, \tag{4·36}$$

and this function possesses a zero at $\zeta = \zeta_0$, so that ξ, which is defined in terms of η by the relation

$$\xi = -\frac{(1+2R_H\zeta)}{mR_H(\zeta - \zeta_0)}\eta, \tag{4·37}$$

is finite at all heights in the liquid. Using (2·9) and (2·10) it then follows that the solutions for u and v are

$$u = -\tfrac{1}{2}\eta\left\{\frac{1+2R_H\zeta}{f+mR_H\zeta}\frac{mJ_m(\beta r)}{\beta r} + J_m'(\beta r)\right\}\cos(m\phi + \sigma t) \tag{4·38}$$

and $$v = \tfrac{1}{2}\eta\left\{\frac{1+2R_H\zeta}{f+mR_H\zeta}J'_m(\beta r)+\frac{mJ_m}{\beta r}\right\}\sin(m\phi+\sigma t). \tag{4·39}$$

It will be noted that $\eta > 0$ in the main part of the liquid if $A > 0$, and thus the velocity field is similar to that in the barotropic case apart from the zonal component v which has altered character near $r = r_0$. We have for P the formula

$$P = \frac{\rho_0 \Omega \mathrm{i}}{\beta m R_H}\{(f+mR_H\zeta)^2-(1+2R_H\zeta)^2\}\frac{\eta}{(\zeta-\zeta_0)},$$

and in order to be consistent with the assumption made concerning λ_1 early in this section, it is necessary to treat the term within the brackets as negative in sign. The complete formula for the pressure is then

$$p = -PJ_m(\beta r)\sin(m\phi+\sigma t), \tag{4·40}$$

so that $m\phi = \frac{1}{2}\pi$ is a line along which p is a maximum and $m\phi = -\frac{1}{2}\pi$ a line along which p is a minimum at $t = 0$.

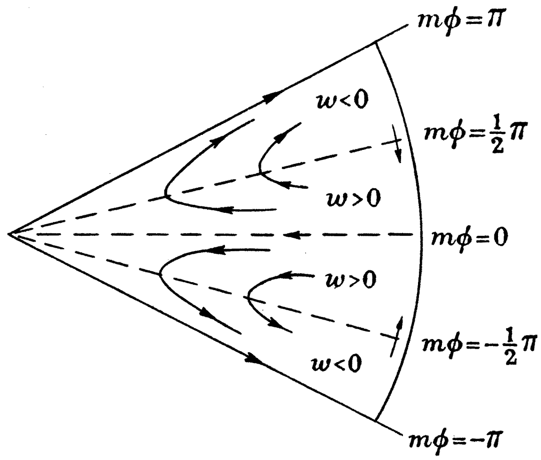

FIGURE 3

Since the temperature function τ is effectively given by $(1/g\alpha)(\mathrm{d}P/\mathrm{d}z)$ it follows that in the main body of the liquid the temperature distribution is such that $m\phi = \frac{1}{2}\pi$ is a line along which the temperature is at a maximum and $m\phi = -\frac{1}{2}\pi$ a line along which the temperature is a minimum. This distribution of temperature is more readily compared with W. H. Dines's statistical results (Brunt 1939) than the barotropic temperature results. The continued association of low pressure at $\zeta = 0$ with high pressure in the main body of liquid above is still in disagreement with these statistical results, and it is unlikely therefore that this can be corrected by any theory which ignores the viscous stresses at the base of the liquid.

It is quite evident that the results which have emerged so far from the modified baroclinic theory are considerably closer to the experiment than the barotropic theory, but it is unlikely that good quantitative agreement can be attained until the singularity ζ_4 is retained in the discussion of the differential equation (2·49). The influence of viscosity and conductivity upon these results is a little obscure and this will be considered now. The differential equation (4·1) can be solved in the form (4·5) when s is given by (2·37) and s' is identically equal to s, that is if $K = R$. It is convenient to make this assumption $K = R$ in order to find out in which direction viscosity and conductivity influence the results. Hence the only change which need be made in (4·9) is to replace the term f/mR_H by

$$-\zeta_1 = \frac{f}{mR_H} - \frac{\mathrm{i}a^2}{2RmR_H}.$$

Thus (4·9) becomes

$$\{1+\zeta_1 a^2 R_v(1-\zeta_1)\}\sin(\mathrm{i}aR_v^{\frac{1}{2}}) - \mathrm{i}aR_v^{\frac{1}{2}}\cos(\mathrm{i}aR_v^{\frac{1}{2}}) = 0, \tag{4·41}$$

where ζ_1 is defined in (2·50). Equation (4·11), which is the second relation between β and f, is changed in the same way in this viscous case. It is now quite evident that the solution for f can be obtained by replacing f in (4·16) and (4·33) by $f - \mathrm{i}a^2/2R$. When this is done the wave forms for which the frequency equation is (4·16) are unstable, since a^2 in this case is negative. On the other hand, those wave forms which have (4·33) as frequency equation are normally damped waves, since a^2 in this case is positive. In this latter case the detailed pattern is as follows:

(a) when $(1+\frac{1}{2}R_H)\dfrac{h}{r_0}R_v^{\frac{1}{2}}x_{sm} \geqslant 2·4$, all waves are damped;

(b) when $(1+\frac{1}{2}R_H)\dfrac{h}{r_0}R_v^{\frac{1}{2}}x_{sm} < 2·4$,

all waves will be damped
one wave is damped, other neutral
one wave is damped, other undamped

if $\dfrac{a^2}{2R} \gtreqless 0·214\left\{2·4 - \dfrac{h}{r_0}R_v^{\frac{1}{2}}x_{sm}(1+\frac{1}{2}R_H)\right\}$.

Interest centres principally upon the neutral wave, since the waves observed in the experiment can persist as long as the requisite steady conditions of temperature and rotation are maintained.

5. The relation between Θ_V and Θ_H

The basic temperature field introduced in (1·15) contains two parameters Θ_V and Θ_H which up to this point have been assumed to be quite independent of one another. In a recent paper Lorenz (1953) has considered this particular point and has shown that Θ_V and Θ_H are in fact linearly related, and it is proposed here to derive this relation by a method which is a slight modification of that due to Lorenz. If a uniqueness result were available for a problem of this type then one might expect such a relation to exist, since the mean horizontal gradient and mean vertical gradient of temperature would both be linearly related to the imposed mean horizontal temperature gradient on the boundary; however, since this approach is not possible we proceed as follows.

We introduce non-dimensional variables into equations (1·1) to (1·6) by writing

$$\left.\begin{aligned} t_1 = t/\Omega, \quad r_1 = r_0 r, \quad z_1 = hz, \quad \phi_1 = \Omega t + \phi, \\ u_1 = r_0\Omega u, \quad v_1 = r_0\Omega v, \quad w_1 = h\Omega w, \\ \rho_1 = \rho_0\rho, \quad T_1 = T_0 + (\Delta T_H)\,T, \quad p_1 = \rho_0 r_0^2\Omega^2 p, \end{aligned}\right\} \tag{5·1}$$

where the original variable has suffix 1 and the unsuffixed symbol is the non-dimensional variable. The equations (1·1) to (1·6) then become

$$\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{1}{\rho R}\left\{\frac{\partial^2 u}{\partial z^2}+\frac{h^2}{r_0^2}\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v}{\partial \phi}\right)\right\}, \tag{5·2}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{uv}{r}=-\frac{1}{\rho r}\frac{\partial p}{\partial \phi}+\frac{1}{\rho R}\left\{\frac{\partial^2 v}{\partial z^2}+\frac{h^2}{r_0^2}\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial u}{\partial \phi}\right)\right\}, \tag{5·3}$$

$$\frac{h^2}{r^2}\frac{\mathrm{d}w}{\mathrm{d}t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{gh}{r_0^2\Omega^2}+\frac{h^2}{\rho r_0^2 R}\left\{\frac{\partial^2 w}{\partial z^2}+\frac{h^2}{r_0^2}\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\}, \tag{5·4}$$

$$\rho=1-\frac{\alpha\Delta T_H}{\rho_0}T, \tag{5·5}$$

$$-\frac{\alpha\Delta T_H}{\rho_0}\frac{\mathrm{d}T}{\mathrm{d}t}+\rho\,\mathrm{div}\,\mathbf{V}=0, \tag{5·6}$$

$$\rho\frac{\mathrm{d}T}{\mathrm{d}t}=\frac{1}{K}\left\{\frac{\partial^2 T}{\partial z^2}+\frac{h^2}{r_0^2}\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}\right)\right\}, \tag{5·7}$$

where ΔT_H is the temperature difference between $z=0$, $r=0$ and $z=0$, $r=1$, and R and K are the Reynolds number and Péclet number already defined in (2·31). These six equations contain the four parameters h/r_0, R, K, $\epsilon=r_0^2\Omega^2/gh$ explicitly, and when the ϵ is combined with $\alpha\Delta T_H/\rho_0$ a fifth parameter enters, namely,

$$R_H^*=g\alpha h(\Delta T_H)/\rho_0 r_0^2\Omega^2. \tag{5·8}$$

From (2·31) it follows that $R_H^*=2R_H$, but it is more convenient to work with R_H^* in the present section. This parameter is the Rossby number of the problem. In the present problem, where h/r_0 is usually considerably less than unity, it may be noted that the neglect of the terms in (5·2) to (5·7) which involve h^2/r_0^2 leads to a consistent set of equations provided the exact viscous and conduction conditions at $r=1$ are suitably relaxed. The neglect of such terms in the equations is the same as saying that the solutions for the velocity, pressure, density and temperature fields are expanded in ascending powers of the parameter h^2/r_0^2, and we retain here only the leading terms of the expansions. This method of procedure exposes the important features of shallow systems, namely, that the vertical gradients of velocity are of considerably greater importance than the horizontal gradients, while the hydrostatic equation is satisfied in the first approximation. The equations (5·2) to (5·7) now become

$$\rho\left(\frac{\mathrm{d}u}{\mathrm{d}t}-\frac{v^2}{r}\right)=-\frac{\partial p}{\partial r}+\frac{1}{R}\frac{\partial^2 u}{\partial z^2}, \tag{5·9}$$

$$\rho\left(\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{uv}{r}\right)=-\frac{\partial p}{r\,\partial \phi}+\frac{1}{R}\frac{\partial^2 v}{\partial z^2}, \tag{5·10}$$

$$\frac{\partial p}{\partial z}=-\frac{gh}{r_0^2\Omega^2}+R_H^*T, \tag{5·11}$$

$$\rho=1-\left(\frac{r_0^2\Omega^2}{gh}\right)R_H^*T, \tag{5·12}$$

$$\left(\frac{r_0^2\Omega^2}{gh}\right)R_H^*\frac{\mathrm{d}T}{\mathrm{d}t}=\rho\,\mathrm{div}\,\mathbf{V}, \tag{5·13}$$

$$\rho\frac{\mathrm{d}T}{\mathrm{d}t}=\frac{1}{K}\frac{\partial^2 T}{\partial z^2}. \tag{5·14}$$

It is now evident that expansions of the velocity vector $\mathbf{V}$, pressure p, density ρ and temperature T are permissible in ascending powers of the parameter R_H^*, since this parameter is not associated with the highest order derivatives. Accordingly, we assume that the complete solution of the above system of equations is expressible in the form

$$\left.\begin{aligned} u &= R_H^* u_1 + R_H^{*2} u_2 + \ldots, & \rho &= 1 + R_H^* \rho_1 + R_H^{*2} \rho_2 + \ldots, \\ v &= r + R_H^* v_1 + R_H^* v_2 + \ldots, & T &= \tau_0 + R_H^* \tau_1 + R_H^{*2} \tau_2 + \ldots, \\ w &= R_H^* w_1 + R_H^{*2} w_2 + \ldots, & p &= p_0 + R_H^* p_1 + R_H^{*2} p_2 + \ldots, \end{aligned}\right\} \tag{5·15}$$

where it is assumed that the terms with zero suffix represent the state of solid rotation, and where all the suffixed quantities are independent of the parameter R_H^*. We now substitute these expansions in the equations (5·9) to (5·14) and equate to zero the coefficients of successive powers of R_H^*.

We have then the following system of equations:

Terms independent of R_H^*:

$$r = \frac{\partial p_0}{\partial r}, \quad 0 = \frac{\partial p_0}{\partial \phi}, \quad -\frac{gh}{r_0^2 \Omega^2} = \frac{\partial p_0}{\partial z}, \quad 0 = \frac{\partial^2 \tau_0}{\partial z^2}. \tag{5·16}$$

These equations give the usual solution for p_0, namely,

$$p_0 = \tfrac{1}{2} r^2 - \frac{gh}{r_0^2 \Omega^2} z,$$

and τ_0 is evidently a linear function of z. If we impose upon T the condition that $\partial T/\partial z$ must vanish at $z = 1$ (see p. 35), which implies that there is no flow of heat across the free surface, then it follows that the only possible form of solution for τ_0 is

$$\tau_0 = \tau_0(r), \tag{5·17}$$

where the right-hand side is a function of r only. This implies that $\partial \tau_0/\partial z$ is zero everywhere in the liquid, and thus there is no heat transfer in the vertical direction associated with this term. A side condition of the form $d\tau_0/dr = 0$ at $r = 1$ can be imposed upon τ_0 in order to ensure that there is no heat flow across the side boundary into the liquid. It may be shown that for a solution of this kind $\partial \tau_2/\partial z$ does not vanish at $z = 0$, and thus the above temperature field will be related to the temperature distribution on $z = 0$. We impose upon $\tau_0(r)$ the condition

$$\tau_0(1) - \tau_0(0) = 1, \tag{5·18}$$

which ensures that the difference in temperature between $z = 0$, $r = 0$ and $z = 0$, $r = 1$ is ΔT_H.

Terms of the first order in R_H:

$$\frac{\partial u_1}{\partial t} - 2v_1 + \epsilon r \tau_0 = -\frac{\partial p_1}{\partial r} + \frac{1}{R} \frac{\partial^2 u_1}{\partial z^2}, \tag{5·19}$$

$$\frac{\partial v_1}{\partial t} + 2u_1 = -\frac{\partial p_1}{r \partial \phi} + \frac{1}{R} \frac{\partial^2 v_1}{\partial z^2}, \tag{5·20}$$

$$\frac{\partial p_1}{\partial z} = \tau_0(r), \tag{5·21}$$

$$\frac{1}{r} \frac{\partial}{\partial r}(r u_1) + \frac{\partial v_1}{r \partial \phi} + \frac{\partial w_1}{\partial z} = 0, \tag{5·22}$$

$$\frac{\partial \tau_1}{\partial t} + u_1 \tau_0'(r) = \frac{1}{K} \frac{\partial^2 \tau_1}{\partial z^2}. \tag{5·23}$$

In order to derive the relation between the mean gradients of temperature in the vertical and in the horizontal it is sufficient at this stage to consider the steady fields, and the terms in $\partial/\partial t$ and $\partial/\partial\phi$ can be ignored for our present purpose. Equation (5·21) can be solved immediately and gives

$$p_1 = z\tau_0(r) + p_1^*(r), \tag{5·24}$$

where $p_1^*(r)$ is a function of r only. We can now use equations (5·19) and (5·20) to determine u_1 and v_1 in terms of τ_0, $p_1^*(r)$, etc., and we obtain

$$2v_1 = \frac{\partial p_1}{\partial r} + \epsilon r\tau_0 - \frac{1}{R}\frac{\partial^2 u_1}{\partial z^2}; \tag{5·25}$$

hence

$$4u_1 = \frac{1}{R}\frac{\partial^2}{\partial z^2}\left\{\frac{\partial p_1}{\partial r} + \epsilon r\tau_0 - \frac{1}{R}\frac{\partial^2 u_1}{\partial z^2}\right\}$$

$$= -\frac{1}{R^2}\frac{\partial^4 u_1}{\partial z^4}. \tag{5·26}$$

The equation of continuity (5·22), in its symmetrical form with $\partial v_1/\partial\phi$ zero, shows that $\partial w_1/\partial z$ satisfies the same equation as u_1. The boundary conditions to be satisfied are $w = 0$ at $z = 0$ and $z = 1$, $u = 0$ at $z = 0$ and $\partial u/\partial z = 0$ at $z = 1$. This problem is precisely the same as that solved in Davies (1953, p. 101), where it is shown that

$$ru_1 = -\frac{\partial\psi_1}{\partial z}, \quad rw_1 = \frac{\partial\psi_1}{\partial r} \tag{5·27}$$

and

$$\begin{aligned}\psi_1 &= F_0(r)\,G_0(z)\\ &= F_0(r)\left\{1 - \frac{\sin\omega z\sinh\omega z}{s_1 S_1} - \frac{\sin\omega(1-z)\sinh\omega(1-z)}{s_1 S_1}\right.\\ &\quad - \frac{S_1^2 s_1 + S_1 C_1 c_1 + s_1 c_1 C_1}{S s_1(S_1 C_1 - s_1 c_1)}\sin\omega z\sinh\omega(1-z)\\ &\quad \left. - \frac{s_1^2 S_1 + c_1 S_1 C_1 + s_1 c_1 C_1}{S_1 s_1(S_1 C_1 - s_1 c_1)}\sin\omega(1-z)\sinh\omega(1-z)\right\}.\end{aligned} \tag{5·28}$$

Here $\omega^2 = R$, $S_1 = \sinh\omega$, $C_1 = \cosh\omega$, $s_1 = \sin\omega$, $c_1 = \cos\omega$ and $F_0(r)$ is an arbitrary function of r. The solution for v_1 is then given by

$$2v_1 = z\tau_0'(r) + \frac{dp_1^*}{dr} + \epsilon r\tau_0 + \frac{1}{Rr}F_0(r)\,G_0'''(z). \tag{5·29}$$

If v_1 is to vanish at $z = 0$ then we must have

$$-\frac{1}{Rr}F_0(r)\,G_0'''(0) = \frac{dp_1^*}{dr} + \epsilon r\tau_0, \tag{5·30}$$

and if $\partial v_1/\partial z$ is to vanish at $z = 1$, then

$$-\tau_0'(r) = \frac{1}{Rr}F_0(r)\,G_0^{\text{iv}}(1). \tag{5·31}$$

These two equations determine p_1^* and $F_0(r)$ in terms of $\tau_0(r)$. Interest centres principally upon $F_0(r)$. It is shown in Davies (1953) that $G_0^{\text{iv}}(1) \sim 4\omega^4 = 4R^2$ when R is sufficiently large, and thus we have

$$F_0(r) = -\frac{1}{4R}r\tau_0'(r). \tag{5·32}$$

Thus when $\tau_0(r)$ is prescribed the stream function ψ_1 is uniquely determined, and we can now proceed to the solution of (5·23) which in the steady case can be written in the form

$$\frac{\partial^2\tau_1}{\partial z^2} = Ku_1\tau_0'(r)$$

$$= K\tau_0'(r)\left\{-\frac{1}{r}F_0(r)\,G_0'(z)\right\}. \tag{5·33}$$

The first integration of this with respect to z leads to

$$\frac{\partial\tau_1}{\partial z} = -\frac{K}{r}\tau_0'(r)\,F_0(r)\,G_0(z) + \Psi(r), \tag{5·34}$$

and, taking $\partial\tau_1/\partial z = 0$ at $z = 1$, it follows that $\Psi(r) \equiv 0$. The function $\partial\tau_1/\partial z$ then vanishes also at $z = 0$, but in the main portion of the liquid lying outside the boundary layers the function $G_0(z)$ has been shown in Davies (1953) to be $1 + o(e^{-\omega z})$, hence in general

$$\frac{\partial\tau_1}{\partial z} = -\frac{K}{r}\tau_0'(r)\,F_0(r),$$

so that, using (5·32),

$$\frac{\partial\tau_1}{\partial z} = \frac{K}{4R}\tau_0'^2(r). \tag{5·35}$$

This result is in general agreement with that obtained by Lorenz. The vertical lapse rate of temperature is now given by

$$\frac{\partial T_1}{\partial z_1} = \frac{\Delta T_H}{h}R_H^*\frac{\partial\tau_1}{\partial z} = R_H^*\frac{\Delta T_H}{h}\left\{\frac{K}{4R}\tau_0'^2(r)\right\}, \tag{5·36}$$

and the horizontal lapse rate of temperature is given by

$$\frac{\partial T_1}{\partial r_1} = \frac{\Delta T_H}{r_0}\tau_0'(r). \tag{5·37}$$

In each of these expressions there will be further terms involving powers of the parameter R_H^* but, since this is small, it is unnecessary to take the investigation further.

Since the horizontal derivatives of velocity and temperature have been ignored with the h^2/r_0^2 terms it is not possible to define the function $\tau_0(r)$ precisely, and we may restate its qualitative properties, namely, $\tau_0'(1) = 0$, $\tau_0(1) - \tau_0(0) = 1$. When the first of these properties is satisfied it will be noted from (5·29) to (5·31) that $v_1(1) = 0$. Likewise $u_1(1) = 0$, and the only condition which cannot be satisfied is $w = 0$ at $r = 1$, although this one is evidently impossible to satisfy from the nature of our approach.

If we compare (5·35) and (5·37) with the assumed temperature distribution (1·15) we see that this assumed basic temperature field is not consistent with the above results, although it must be borne in mind that they refer to a mean state in the body of the liquid and well away from the boundaries. It is sufficient here to compare (5·36), (5·37) and (1·15) in their average values. Hence if we use a bar to denote an average value of a quantity over the range $r = 0$ to $r = 1$, so that

$$\bar{\phi} = \int_{r=0}^{r=1}\phi(r)\,r\,\mathrm{d}r, \tag{5·38}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

it then follows that
$$\frac{\Theta_V}{\frac{1}{2}r_0\,\Theta_H} = \frac{\partial \overline{T_1}}{\partial z_1}\bigg/\frac{\partial \overline{T_1}}{\partial r_1} = \frac{r_0}{h}R_H^*\frac{K}{4R}\frac{\overline{\tau_0'^2(r)}}{\overline{\tau_0'(r)}},$$

and hence
$$\frac{\Theta_V}{h\Theta_H} = \frac{1}{8}\frac{r_0^2}{h^2}R_H^*\frac{K}{R}\frac{\overline{\tau_0'^2(r)}}{\overline{\tau_0'(r)}}. \tag{5·39}$$

Using the definitions of R_H and R_v in (2·31) and noting that $R_H^* = 2R_H$, it now follows that

$$\frac{h^2R_v}{r_0^2} = \tfrac{1}{4}R_H^2\frac{K}{R}\frac{\overline{\tau_0'^2(r)}}{\overline{\tau_0'(r)}}, \tag{5·40}$$

and thus we see that the parameters $hR_v^{\frac{1}{2}}/r_0$ and R_H are on the average related linearly. This is the precise result which is needed to bring the present theory in line with the experiments. It remains now to make quantitative comparisons.

If we choose $\tau_0'(r) = 4(r-r^3)$ and $\tau_0(r) = 2r^2-r^4$, then it is easily shown that $\overline{\tau_0'(r)} = \frac{8}{15}$, $\overline{\tau_0'^2(r)} = \frac{2}{3}$, so that
$$\frac{h^2R_v}{r_0^2} = \tfrac{5}{16}R_H^2\frac{K}{R}. \tag{5·41}$$

A more precise choice of $\tau_0'(r)$ can be made by fitting a curve to the values of V_0 given in table 1, since V_0 is proportional to the radial temperature gradient, but this makes a negligible change in the result (5·41). Since $K/R = \mu c_v/k = \sigma =$ Prandtl number, we have, using the value $\sigma = 7$ which is valid for water at a temperature of about 18° C,

$$\frac{h^2R_v}{r_0^2} = 2{\cdot}2R_H^2,$$

and hence
$$\frac{hR_v^{\frac{1}{2}}}{r_0} = 1{\cdot}5R_H, \tag{5·42}$$
approximately.

It must be emphasized that the results (5·41) and (5·42) have been derived on the assumption that the flow is symmetrical about the axis. When there are waves present, it is most probable that the mean temperature fields will be altered; in the few experimental temperature distributions which are available to the author this is certainly true. Thus it is likely that the application of (5·42) to the stability formula (4·29) will produce results which are likely to be in increasing error as m, the wave number, increases. The determination of the modified formula which replaces (5·41) in the case of asymmetry is a difficult problem and one which will not be discussed any further here.

6. Comparison of theoretical and experimental results

If we substitute for $hR_v^{\frac{1}{2}}/r_0$ from (5·42) in (3·17) we obtain the following barotropic formula for the velocity of propagation of waves:

$$\frac{\omega}{\Omega} = 0{\cdot}9R_H \tag{6·1}$$

approximately, and this has to be compared with Hide's result (3·24). It is clear that this barotropic formula gives no indication of any instability and gives a result which is about seven times too large for the angular velocity of the waves.

If we use the baroclinic results we obtain, from (4·29),

$$x_{sm} R_H (1+\tfrac{1}{2}R_H) < 1{\cdot}6 \tag{6·2}$$

as the criterion for instability. This criterion is then similar in form to that obtained from the experiments; the numerical results are given in table 4. No detailed experimental results are available to check table 4, but the results vary in the right direction and are substantially of the correct order of magnitude. These results refer expressly to the problem in which the liquid occupies the region $0 \leqslant r \leqslant r_0$. Experimental results for the annulus case in which the liquid occupies a region $r_i \leqslant r \leqslant r_0$, with $r_i = 2{\cdot}45$ cm and $r_0 = 4{\cdot}90$ cm, have been kindly supplied by Fultz in a private communication, and it is worth while at this stage to derive the theoretical results in this case.

Table 4

wave number, m	0	1	2	3	4	5	6
value of R_H when m-wave becomes unstable	0·56	0·40	0·29	0·24	0·20	0·17	0·15

It may easily be verified that all the analysis of § 4 applies to the annulus case provided the quantity x_{sm} in (4·18) which appears as the sth zero of $J_m(x) = 0$ is suitably modified. It is necessary to replace x_{sm} by $r_0\beta_{sm}$, where $\beta = \beta_{sm}$ is the sth zero of the equation

$$J_m(\beta r_0)\, Y_m(\beta r_i) - J_m(\beta r_i)\, Y_m(\beta r_0) = 0, \tag{6·3}$$

and with this definition of β_{sm} the instability criterion (4·29) becomes

$$h\beta_{sm} R_v^{\frac{1}{2}} (1+\tfrac{1}{2}R_H) < 2{\cdot}4. \tag{6·4}$$

As before, interest centres upon the first zero β_{1m}, and this can be determined with sufficient accuracy by substituting in (6·3) the asymptotic expansions of J_m and Y_m. This leads to the result

$$\beta_{1m} = \frac{\pi}{r_0 - r_i}\left\{1 + \frac{(4m^2-1)\,(r_0-r_i)^2}{8\pi^2 r_0 r_i} + \dots\right\}. \tag{6·5}$$

The methods used in § 5 apply to the annulus case except that the averaging process given in (5·38) is now defined as follows:

$$\overline{\phi} = \int_{(r_i/r_0)}^{1} \phi(r)\, r\, \mathrm{d}r, \tag{6·6}$$

while $\tau_0(r)$ is a function satisfying $\tau_0'(1) = \tau_0'(r_i/r_0) = 0$ and $\tau_0(1) - \tau_0(r_i/r_0) = 1$. Writing $x = r_i/r_0$ and choosing $\tau_0'(r) = 12(r-x)\,(1-r^2)/(1-x)^3\,(3+x)$, it follows that the appropriate generalization of the result (5·41) is

$$\frac{h^2 R_v}{r_0^2} = \tfrac{3}{14} R_H^2 \frac{K}{R} \frac{(1-x)\,(35+47x+25x^2+5x^3)}{(3+x)\,(8-7x+3x^2+3x^3+3x^4)}. \tag{6·7}$$

Combining (6·4) and (6·7) we obtain the generalization of (6·2) to the annulus case.

When $r_i = \frac{1}{2}r_0 = 2{\cdot}45$ cm, which is the case investigated by Fultz, (6·4) becomes

$$R_H(1+\tfrac{1}{2}R_H) < \frac{1{\cdot}6}{\beta_{sm} r_0}, \tag{6·8}$$

taking $K/R = 7$ as for (5·42). Fultz uses the Rossby number Ro_T^* which is related to R_H by

$$Ro_T^* = (1+x)\, R_H, \tag{6·9}$$

and table 5 can then be constructed of the critical Ro_T^* values. It will be observed from table 5 that the agreement between the theoretical and experimental results is fairly satisfactory for small values of m but when $m \geqslant 3$ the divergence of the results is quite considerable. The wider divergence of the results when m increases can partly be explained by the neglect of the singularities ζ_3 and ζ_4 in building up the present baroclinic theory, and this will be enlarged upon shortly. Part of the discrepancy also is due to the approximate character of the relation (5·42), since the term on the right-hand side of this relation is merely the leading term of an infinite series. It is also necessary to note that the experimental values are obtained by keeping all the physical parameters fixed except the temperature difference $\Delta\theta$ between the two cylinders which is steadily increased from $\Delta\theta = 0$. Fultz states that the transition values will be different if $\Delta\theta$ is steadily decreased to zero; thus a given wave number can exist over a range of values of Ro_T^*. For example, in the experiment described, it is found that the upper limits of these ranges for wave numbers 4 and 5 are respectively 0·18 and 0·08 (cf. theoretical values 0·16 and 0·13), and it is possible that the theoretical values should be compared with this upper limit. The existence of this range, a phenomenon which has been called 'metastability' by Fultz, is not explained by any of the theory up to this point, and it is useful to conclude the present section with a brief discussion of the general stability characteristics of the flow.

TABLE 5. ANNULUS CASE $r_i = \frac{1}{2}r$

(Mean temperature 21° C, $\alpha = 2{\cdot}1 \times 10^{-4}$)

wave number, m	0	1	2	3	4	5
critical theoretical values of Ro_T^*	0·28	0·24	0·22	0·18	0·16	0·13
critical experimental values of Ro_T^*	0·26	0·22	0·20	0·11	0·07	0·04

The instability which is associated with the criterion (6·2) is linked with the appearance of a singular point $\zeta = \zeta_1$ within the range $0 < \zeta < 1$. At this point the function s, defined in (2·37), vanishes, and thus the angular velocity of the wave system equals the angular velocity of the liquid at some height within the liquid (it may be recalled that the effective angular velocity of the basic flow varies linearly with height). This type of instability accordingly is of the same type as that investigated by Heisenberg, Lin, Meksyn and others in the two-dimensional liquid motion between parallel planes, where the wave speed c becomes equal to the stream speed U at some intermediate point within the liquid.

The differential equation (2·49) has singular points also at the zeros $\zeta = \zeta_3$ and $\zeta = \zeta_4$ of λ_1, these being defined precisely in (2·52) and (2·53), and as stated earlier ζ_3 will lie outside the range $0 < \zeta < 1$, but ζ_4 may lie inside this range when R is effectively infinite. The function λ_1 is the determinant of the two equations of horizontal motion, and the vanishing of λ_1 implies that the horizontal velocity components are infinitely large. That this is so also follows from (2·49) when we investigate the solution in the neighbourhood of $\zeta = \zeta_4$. Making R infinite, we have

$$\zeta_4 = (1-f)/R_H(m-2),$$

and, bearing in mind that $f \ll 1$, it is evident that $m = 0$ and $m = 1$ make ζ_4 negative, so that there is no possibility of any resonance of the horizontal field in this case, likewise $m = 2$ makes ζ_4 infinite, and thus resonance of the horizontal velocity can occur only when $m > 2$. The effect of the ζ_4 singularity is discussed in the next section.

7. The variational method of solving the eigenvalue problem

The problem which is presented by equation (2·48) and the associated boundary conditions $\overline{W} = 0$ at $\zeta = 0$ and $\zeta = 1$ can be considered as a problem in the calculus of variations provided that the aim is merely to determine the wave-velocity formula and the stability criterion. In order to do this it is first of all necessary to write (2·48) in the Sturm–Liouville form, namely,

$$\frac{d}{d\zeta}\left\{k(\zeta)\frac{d\overline{W}}{d\zeta}\right\}+\{\lambda^* g(\zeta)-l(\zeta)\}\,\overline{W}=0, \tag{7·1}$$

and it is convenient, for the present purpose, to consider R_v to be the unknown eigenvalue in (2·48) so that

$$\lambda^* = R_v, \tag{7·2}$$

and $k(\zeta)$, $g(\zeta)$, $l(\zeta)$ are functions of ζ, not containing the parameter λ^*, which can be expressed in terms of s, s', λ_1, etc., in (2·48). When the transformation to the form (7·1) is effected it then follows Courant & Hilbert (1953) that the minimum value of the integral

$$I=\int_0^1\left\{k(\zeta)\left(\frac{d\overline{W}}{d\zeta}\right)^2+l(\zeta)\,\overline{W}^2(\zeta)\right\}d\zeta, \tag{7·3}$$

subject to the normalizing condition

$$\int_0^1 g(\zeta)\,\overline{W}^2(\zeta)\,d\zeta=1 \tag{7·4}$$

is λ^*. This minimum is attained when $\overline{W}(\zeta)$ is the exact solution of (7·1) which satisfies $\overline{W} = 0$ at $\zeta = 0$ and $\zeta = 1$. An approximate value of λ^* can be obtained if a function $\overline{W}_A(\zeta)$ is taken which is an approximation to the exact $\overline{W}(\zeta)$.

We consider first the barotropic problem discussed in § 3 in order to assess the degree of error in this approach to the problem. In the barotropic problem, by comparing (7·1) and (3·1) with f^2 neglected compared with 1 or a^2R_v, it is easily shown that

$$k(\zeta) = -\frac{g(\zeta)}{a^2} = \exp-(mR_H\zeta/f), \quad l(\zeta) = 0. \tag{7·5}$$

In order to construct the approximate $\overline{W}$ function in this case it may be noted that $\zeta = 0$ and $\zeta = 1$ are simple zeros of $\overline{W}$. When $\overline{W}$ vanishes (3·1) indicates that the second differential at such a point becomes large when f is small; near $\zeta = 0$ it is easily shown that $\overline{W}$ is of the form $\zeta\exp(\zeta mR_H/2f)$, hence we assume that an approximation for $\overline{W}$ will be given by

$$\overline{W} = C\zeta(1-\zeta)\exp(mR_H\zeta/2f), \tag{7·6}$$

where C is a constant. Equation (7·4) then gives the following relation for C:

$$-a^2C^2\int_0^1 \zeta^2(1-\zeta)^2\,d\zeta = 1, \tag{7·7}$$

and the approximate value of R_v will be given by

$$R_v = C^2\int_0^1\left\{1-2\zeta+\frac{mR_H}{2f}\zeta(1-\zeta)\right\}^2 d\zeta. \tag{7·8}$$

By eliminating C between (7·7) and (7·8) we obtain

$$a^2R_v\int_0^1 \zeta^2(1-\zeta)^2\,d\zeta+\int_0^1\left\{1-2\zeta+\frac{mR_H}{2f}\zeta(1-\zeta)\right\}^2 d\zeta = 0, \tag{7·9}$$

and upon integration this gives

$$a^2R_v+\frac{m^2R_H^2}{4f^2}+10 = 0. \tag{7·10}$$

This must be compared with (3·5), with $n = 1$, and it will be observed that the only departure is in the final term where 10 has replaced π^2 $(= 9{\cdot}8696)$. Thus the error is about $1\frac{1}{2}$ %. This method gives of course the first eigenvalue only, but in the present problem it is only the first eigenvalue which is of any interest.

We consider now the general case, and by comparing (7·1) and (2·49) we have

$$\left.\begin{aligned} k(\zeta) &= \lambda_1 s^{-1}(\zeta-\zeta_2)^{-a_2}(\zeta-\zeta_3)^{-a_3}(\zeta-\zeta_4)^{-a_4},\\ g(\zeta) &= -a^2\frac{sk(\zeta)}{\lambda_1 s'},\\ l(\zeta) &= a^2\frac{s^2}{\lambda_1}k(\zeta), \end{aligned}\right\} \tag{7·11}$$

where s, s' and λ_1 are defined in (2·37), (2·38) and (2·46) and a_2, a_3, a_4 are given by

$$a_2 = (1+2R_H\zeta_2)\bigg/\left\{\left(1-\frac{2f}{m}+\frac{ia^2}{Km}\right)^2-\left(\frac{a^2}{2R}-\frac{a^2}{2K}\right)^2\right\},$$

$$a_3 = m\bigg/\left\{m-2f-\tfrac{1}{2}ia^2\left(\frac{m}{R}-\frac{m+2}{K}\right)\right\},$$

$$a_4 = m\bigg/\left\{m-2f+\tfrac{1}{2}ia^2\left(\frac{m}{R}-\frac{m-2}{K}\right)\right\}.$$

We proceed upon the assumption that $K = R$, so that we have

$$a_2 = a_3 = a_4 = m\bigg/\left\{m-2f+\frac{ia^2}{R}\right\}; \tag{7·12}$$

hence in this case we have, apart from an arbitrary constant factor which is of no consequence here,

$$\left.\begin{aligned} k(\zeta) &= \lambda_1^{1-a_2}s^{-1-a_2},\\ g(\zeta) &= -a^2\lambda_1^{-a_2}s^{-1-a_2},\\ l(\zeta) &= a^2\lambda_1^{-a_2}s^{1-a_2}, \end{aligned}\right\} \tag{7·13}$$

since $s = s'$ when $K = R$. Ultimately, in order to obtain the wave-velocity formula, R will be made to tend to infinity in which case a_2 tends to $m/(m-2f)$, and thus for the west-east moving waves $(f<0)$, it follows that $0<a_2<1$; in this case s vanishes at $\zeta = \zeta_0 = -f/mR_H$, and it is evident that some difficulties are going to arise in the integration through $\zeta = \zeta_0$. The approximating function for $\overline{W}$ will be chosen to be simply

$$\overline{W} = C\zeta(1-\zeta), \tag{7·14}$$

since this satisfies the boundary conditions, and it is clear from the nature of the problem and also from the barotropic and baroclinic solutions already investigated that $\overline{W}$ has no singularities or zeros at any point of the range $0<\zeta<1$. From (7·3) it is clear that we now have the following integrals to evaluate:

$$I_1 = \int_0^1 \lambda_1^{1-a_2} s^{-1-a_2}(1-2\zeta)^2\,d\zeta, \tag{7·15}$$

$$I_2 = \int_0^1 \lambda_1^{-a_2} s^{1-a_2}\zeta^2(1-\zeta)^2\,d\zeta, \tag{7·16}$$

$$I_3 = \int_0^1 \lambda_1^{-a_2} s^{-1-a_2}\zeta^2(1-\zeta)^2\,d\zeta, \tag{7·17}$$

and it then follows that the required approximate eigenvalue for $\lambda^* = R_v$ is given by

$$a^2 R_v I_3 + I_1 + a^2 I_2 = 0. \tag{7·18}$$

In order to clear up certain integration difficulties it is useful to repeat here the modified baroclinic problem which has been discussed in detail in § 4, and this will serve to indicate also to what extent (7·14) is a reasonable approximation to $\overline{W}$. In this case we take $\lambda_1 = 1$, $a_2 = 1$ and the I_2 term is completely ignored; this then arises from the following differential equation:

$$\frac{d}{d\zeta}\left\{s^{-2}\frac{d\overline{W}}{d\zeta}\right\} - a^2 R_v s^{-2}\overline{W} = 0, \tag{7·19}$$

which is identical with (4·1). With the same assumptions the integrals to be evaluated are

$$I_1 = \int_0^1 s^{-2}(1-2\zeta)^2\,d\zeta, \quad I_3 = \int_0^1 s^{-2}\zeta^2(1-\zeta)^2\,d\zeta, \tag{7·20}$$

where s is given by
$$s = imR_H\left(\zeta + \frac{f}{mR_H} - \frac{ia^2}{2R}\right) = imR_H\left(\zeta - \zeta_0 - \frac{ia^2}{2R}\right), \tag{7·21}$$

and $0<\zeta_0<1$. Provided we retain the term $ia^2/2R$ in s no difficulty arises in the integration, since the zero of s then lies outside the line of integration. When the integration is performed in this way and we subsequently make $R\to\infty$ we obtain

$$-m^2R_H^2 I_1 = 8 - \frac{1}{\zeta_0(1-\zeta_0)} + 4(2\zeta_0 - 1)\left\{\ln\left(\frac{1-\zeta_0}{\zeta_0}\right) + i\pi\right\}, \tag{7·22}$$

$$-m^2R_H^2 I_3 = \tfrac{1}{3} - 4\zeta_0 + 4\zeta_0^2 + 2\zeta_0(1-\zeta_0)(1-2\zeta_0)\left\{\ln\left(\frac{1-\zeta_0}{\zeta_0}\right) + i\pi\right\}. \tag{7·23}$$

If for the present we ignore the $i\pi$ terms in (7·22) and (7·23), it follows from (7·18) that the required approximate relation between R_v and ζ_0 is then

$$a^2R_v\left\{\tfrac{1}{3} - 4\zeta_0 + 4\zeta_0^2 + 2\zeta_0(1-\zeta_0)(1-2\zeta_0)\ln\left(\frac{1-\zeta_0}{\zeta_0}\right)\right\} = \frac{1}{\zeta_0(1-\zeta_0)} - 8 - 4(2\zeta_0-1)\ln\left(\frac{1-\zeta_0}{\zeta_0}\right), \tag{7·24}$$

since I_2 is being ignored in this case. This relation has to be compared with (4·19) which we can write in the form

$$1 + a^2R_v\zeta_0(1-\zeta_0) = aR_v^{\frac{1}{2}}\coth aR_v^{\frac{1}{2}}. \tag{7·25}$$

It may be recalled that (7·25) is the exact relation arising from (7·19). The curve of $aR_v^{\frac{1}{2}}$ against ζ_0 has been given in figure 3, and the principal features of this curve are its symmetry about the line $\zeta_0 = \frac{1}{2}$, $aR_v^{\frac{1}{2}}$ tends to infinity as ζ_0 tends to 0 or 1, and $aR_v^{\frac{1}{2}} = 2·40$ when $\zeta_0 = \frac{1}{2}$. The curve of $aR_v^{\frac{1}{2}}$ against ζ_0 given by (7·24) has similar features, but when

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

$\zeta_0 = \frac{1}{2}$ we now have $aR_v^{\frac{1}{2}} = 6^{\frac{1}{2}} = 2{\cdot}45$. Thus, provided the $i\pi$ terms occurring in (7·22) and (7·23) are ignored it is possible to achieve a similar result to the exact one, using (7·14), which gives an error of about 2 %.

It may be noted before proceeding any further that the results (7·21) and (7·22) for the integrals I_1 and I_3 could be obtained also by making R tend to infinity initially, and in the integration process we then avoid the singularity at $\zeta = \zeta_0$ by introducing a semi-circle $\Gamma_2 : \zeta = \zeta_0 + \epsilon\, e^{i\theta}$ $(\pi \leqslant \theta \leqslant 2\pi)$, which has its centre at $\zeta = \zeta_0$ and the path of integration from $\zeta_0 - \epsilon$ to $\zeta_0 + \epsilon$ is replaced by Γ_2. The $i\pi$ terms in (7·22) and (7·23) then arise from the integration around Γ_2 when ϵ tends to zero. As we have noted above, in order to achieve the same result as in § 4 it is necessary to discard the $i\pi$ terms; this is equivalent to taking the 'finite part' of the infinite integrals (7·20) so that we effectively take the *mean value* of two integrals to achieve the correct answer, and these integrals differ only in that the semicircle around the singularity at $\zeta = \zeta_0$ is above the line in one case and below the line in the other. A rigorous justification of this procedure is possible.

We are now in a position to extend the results of this paper beyond the baroclinic investigation of § 4, and this will be done by investigating the integrals (7·15), (7·16) and (7·17) in greater detail. The exact evaluation of these integrals is not possible in any simple form, and it is necessary to consider simplified forms of the integrands which will emphasize the different parameters of the problem. The modified baroclinic investigation of § 4 corresponds to $\lambda_1 = 1$, $a_2 = 1$ and I_2 being neglected. A case which is readily investigated is that when $\lambda_1 = 1$, $a_2 = m/(m-2f)$ and I_2 is retained. The retention of I_2 is equivalent to the retention of the term ss' in $(R_v + ss')\,\overline{W}$ in (2·49). The result is, however, not significantly different from that given in (7·24) and will therefore be omitted. It has been pointed out in § 6 that the presence of the singularity at $\zeta = \zeta_4$ may have an important bearing upon the results of § 4, and the remainder of this section will be devoted to an investigation of the case in which this singularity is retained. For simplicity we now choose $\lambda_1 = 1 - (\zeta/\zeta_4)$, where ζ_4 is given approximately by $\zeta_4 R_H(m-2) = 1$ (see (2·53)); we choose $a_2 = 1$ and I_2 will be neglected. We then have to evaluate

$$I_1 = \int_0^1 s^{-2}(1-2\zeta)^2\, d\zeta, \tag{7·26}$$

$$I_3 = \int_0^1 s^{-2}\zeta^2(1-\zeta)^2\lambda_1^{-1}\, d\zeta, \tag{7·27}$$

where s, as before, is given by (7·21) in the general case. It will be noted that as ζ_4 tends to infinity this case becomes identical with that investigated in (7·20) et seq. The integral I_1 has been evaluated in (7·21) and we find that I_3 is given by

$$-m^2R_H^2 I_3 = \zeta_4\left\{P \ln\left(\frac{1-\zeta_0}{\zeta_0}\right) + Q \ln\frac{\zeta_4}{|\zeta_4 - 1|} + R\right\}, \tag{7·28}$$

where
$$P = \frac{\zeta_0^2(1-\zeta_0)^2}{(\zeta_4-\zeta_0)^2} + \frac{2\zeta_0(1-\zeta_0)(1-2\zeta_0)}{(\zeta_4-\zeta_0)}, \tag{7·29}$$

$$Q = \left\{\frac{\zeta_4(1-\zeta_4)}{\zeta_4-\zeta_0}\right\}^2, \tag{7·30}$$

$$R = \frac{1}{(\zeta_4-\zeta_0)}\{3\zeta_0^2 - \tfrac{5}{2}\zeta_0 + \tfrac{3}{2}\zeta_4 - \zeta_4\zeta_0 - \zeta_4^2\}, \tag{7·31}$$

and the 'finite part' has again been chosen. The approximate relation between R_v and the remaining parameters is $a^2R_vI_3+I_1=0$, where I_1 is taken in the form

$$-m^2R_H^2I_1=-\frac{1}{\zeta_0(1-\zeta_0)}+8+4(2\zeta_0-1)\ln\left(\frac{1-\zeta_0}{\zeta_0}\right). \tag{7·32}$$

As in the previous case we see that as ζ_0 tends to zero or unity I_1 tends to ∞, while I_3 tends to a finite limit, thus $aR_v^{\frac{1}{2}}\to\infty$ as $\zeta_0\to 0$ or $\zeta_0\to 1$, just as in figure 3, while $aR_v^{\frac{1}{2}}$ attains a minimum value at an intermediate value of ζ_0. Since $\zeta_4=1/(m-2)\,R_H$ it follows that the experimental values of ζ_4 will be fairly large, and when ζ_4 is assumed to be large it may be shown that the minimum value of $aR_v^{\frac{1}{2}}$ is attained where $\zeta_0=\frac{1}{2}-\frac{1}{12\zeta_4}$ and we have

$$(aR_v^{\frac{1}{2}})_{\text{min.}}=2{\cdot}4\left(1-\frac{1}{4\zeta_4}\right). \tag{7·33}$$

It then follows that the criterion (6·2) becomes changed approximately to

$$x_{sm}R_H(1+\tfrac{1}{2}R_H)<1{\cdot}6\left(1-\frac{1}{4\zeta_4}\right),$$

and inserting the approximate value $\zeta_4=1/(m-2)\,R_H$ this becomes

$$x_{sm}R_H(1+\tfrac{1}{4}mR_H)<1{\cdot}6. \tag{7·34}$$

Thus for $m=5$, $x_{1m}=8{\cdot}8$ and the critical value of R_H for instability is $R_H=0{\cdot}12$ compared with $R_H=0{\cdot}13$ in table 4, and it is evident that the results which are derived from (7·34) are nearer the experimental figures quoted in this table (for $m\geqslant 2$), although it must be borne in mind that the latter refer to an experiment in which the liquid is bounded radially by two cylinders, while (7·34) applies to a liquid which is bounded externally by one cylinder only.

8. Conclusion

It is quite clear that the present theory establishes the inadequacy of barotropic theory in explaining stability phenomena in the dishpan experiment and shows that baroclinic theory, as yet in an approximate form, is capable of giving the main trend of the stability results. A more detailed investigation of equation (2·49) is essential to obtain more accurate quantitative results. However, even when this is accomplished the problem of incorporating the vertical acceleration term $w\,\partial V_0/\partial z$ remains, and it would seem that the phase changes of the long waves in the vertical direction (which do not appear in the present solution except when f is a complex quantity and the waves are growing or decreasing in amplitude) are intimately connected with this term. Much of the theory can be carried over into the corresponding atmospheric problem, and it throws some light upon the changes which occur in the long-wave patterns of the upper part of the troposphere.

References

Brunt, D. 1939 *Physical and dynamical meteorology*. Cambridge University Press.

Courant, R. & Hilbert, D. 1937 *Methoden der Mathematischen Physik*, vol. **1**, 2nd ed. Berlin: J. Springer.

Davies, T. V. 1953 *Phil. Trans.* A, **246**, 81–112.

Fultz, D. 1953 *Symposium on model experiments*. Department of Civil Engineering, Johns Hopkins University, Baltimore, Maryland.

Hide, R. 1953 Unpublished Thesis, University of Cambridge.

Kuo, H. L. 1954 *J. Met.* **4**, 399–411.

Lorenz, E. N. 1953 *Symposium on model experiments*. Department of Civil Engineering, Johns Hopkins University, Baltimore, Maryland.

Starr, V. P. & Long, R. R. 1953 *Geophys. Res. Pap.* no. 24, pp. 103–113.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES